The Nash problem

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Za mamu i tatu.

Preface

This thesis is the culmination of two years' study at the Department of Mathematical Sciences of the University of Copenhagen. Particularly over the last six months, I have had the pleasure of learning some very beautiful mathematics. I hope that with this thesis I have done justice to what I have learned.

Acknowledgements

First and foremost I would like to thank my supervisor Lars Halvard Halle for introducing me to this wonderful area of mathematics, and for the incalculable number of hours he has invested into our regular discussions over the last six months. I am very grateful.

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To my two flatmates Adrian Eberhard and Tim Selbach, a huge thank you for your impressive levels of tolerance, and for keeping me fed during my degree. You have been very supportive, but most importantly it is just great fun living with you guys.

Thanks are also due for Adrien Vakili, who reminded me of the importance of beautiful typesetting. Any percieved beauty in this text is much to his credit. He has given me some valuable advice regarding the dark arts of LaTeX as well.

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Ovaj magistarski posvećujem svojim roditeljima Tamari i Samiru. Jako mi je drago što ste uvijek vjerovali u mene, i što ste mi pomagali sa svim problemima koje sam imao. Ozbiljno, ne bih nikad uspio nešto ovako bez vaše pomoći, i zbog toga sam se pogotovo trudio za vas. Vi ste moji heroji. Nadam se da ste zadovoljni vašim sinom.

Copenhagen, July 2015 Alvin Šipraga

Contents

Introduction

The principal objective of this thesis is to give a rigorous and largely self-contained introduction to the Nash problem, including the requisite theory of arcs.

Overview of the Nash problem

Resolution of singularities is one of the main devices used in the study of singular varieties. Roughly speaking, a resolution of a singular variety X is a proper birational morphism $Y \to X$ from a smooth variety Y, and in this case we say that X and Y are birational. Birational varieties are known to share many common properties, and the idea is to take a resolution so that these properties may be examined on a more well-behaved smooth variety instead.

The caveat is that there is often no obvious choice of resolution, and so the task of extracting meaningful information about the singular variety is still quite complicated. In this way, the problem becomes one of classification of resolutions and their mutual properties. Unsurprisingly, more can then be said about the singular variety than if the focus was on just one possible resolution.

As part of this effort to better understand resolutions, the notion of *essential components* was introduced by John F. Nash, Jr. in a 1968 preprint—later published as [Nas95]. Since then the theory has matured considerably, and it is usually more intuitive to speak of *essential divisors* instead. A brief description of these two ideas now follows.

Informally, an essential divisor is an exceptional divisor that appears on every possible resolution of a given variety X. Then for a resolution $Y \to X$, the essential components are the images on Y of the exceptional divisors. In particular, the essential components are all irreducible components of the preimage of the singular locus of X. For every resolution of X, the essential divisors are in bijective correspondence with the essential components. So by definition, the essential divisors depend only on the singular variety X, and not on any particular resolution.

Nash's idea was to employ the notion of *arcs* to study the the essential divisors. One can think of an arc on X as the infinitesimal analogue of an element of the tangent space at a point on X—indeed, the arcs are naturally constructed according to this very intuition. Then by considering the collection of all arcs on X, the so-called *arc space* is obtained, and it turns out that the arc space possesses a natural scheme structure. This highly geometrical phenomenon helps in making the theory an extremely powerful piece of algebro-geometric weaponry.

The modern theory of arcs has undergone a considerable evolution since the time of Nash's original preprint, and is today largely scheme-theoretic. The result is a beautiful piece of mathematical machinery that can operate in a very general setting. In particular, it is possible to define arc spaces on nearly arbitrary schemes. Remarkably little is lost through this generalisation.

Formally, an arc on X is defined to be a morphism (of schemes) of the form $\operatorname{Spec} K[[t]] \to X$, where K is a field extension of the base field k of X. A somewhat peculiar aspect of arcs that distinguishes them from their finite precursors—known as *jets*—is that they each can map to *two* points on X rather than just one. The image of the closed point of $\operatorname{Spec} K[[t]]$ is then said to be the *center* of the arc. This provides a kind of mechanism with which to measure X.

On this basis, Nash focused on the set of arcs centered anywhere on the singular locus of X. Treated as a subset of the arc space, he took the irreducible components of this set and discarded those which did not contain an arc with image partly in the smooth locus of X. He called the remaining components *arc families*, although today arc families are commonly known as *Nash components*. Finally, he constructed a map

$$\mathcal{N} : \{ \text{ Nash components } \} \to \{ \text{ essential divisors } \}$$

and showed it to be injective. Today, the map $\mathcal{N}$ is known as the *Nash map*.

The genesis of the so-called *Nash problem* is the following question posed in [Nas95, p. 36]:

> Here a question arises. Is there always a corresponding arc family for an essential component of this class? And, in general, how completely do the essential components correspond to the arc families?

More succinctly, the Nash problem asks: Is the Nash map bijective?

Things have come a long way since 1968. Proofs have been offered for numerous classes of variety, using a rather diverse range of methods. The problem on varieties of a particular dimension seems to be commonly perceived as the most important case, and until recently this was still an open problem. The following table chronicles recent developments.

Recent answers to the Nash problem

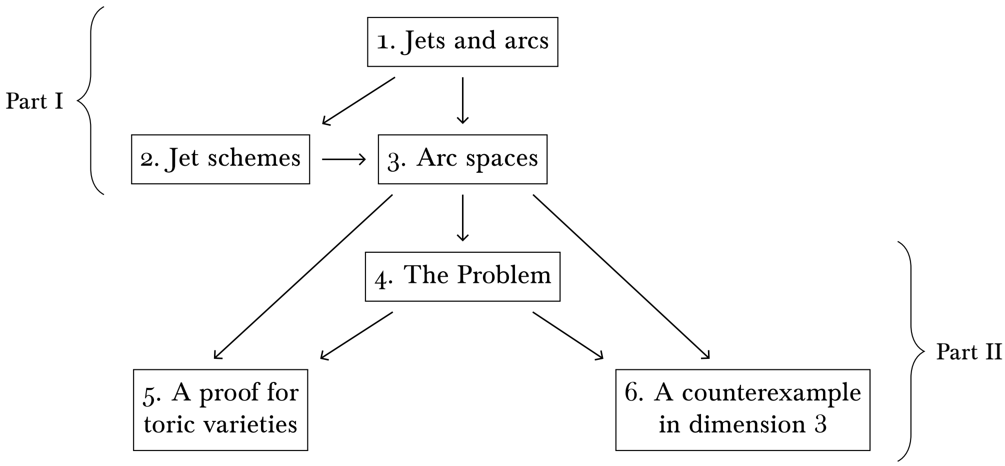
Authors	Answer	Dimension	Characteristic	Paper
Ishii and Kollár	false	≥ 4	$\neq 2, 3$	[IK03]
F. de Bobadilla and Pereira	true	2	0	[FdBP12]
de Fernex	false	≥ 3	0	[dF13]

In light of these results, a *revised Nash problem* was proposed in [JK13, Problem 38] that attempts to compensate for the techniques used in existing counterexamples. The new problem asks whether the Nash map is surjective onto the set of *very essential divisors*, which—as the name suggests—are not as general as essential divisors. In the same paper, it is pointed out that the Nash map is still injective in this case, and therefore existing positive results can still be seen as answers to this new problem.

A comprehensive survey of the Nash problem is given in [PS15].

Structure of the text

The text is divided into two parts. In the first part the theory of arcs is built up from scratch, starting with the notion of jets and jet schemes. The second part then describes the Nash problem and examines two significant results. The relationship between respective chapters is illustrated below.



Chapter 1 gives a brief description of arcs—the basic object used throughout the rest of the text. Jets are also introduced.

Chapter 2 lays the foundational theory of jet schemes, from which the existence of arc spaces is later proved.

Chapter 3 introduces the notion of an arc space. Their existence is proved, along with a number of fundamental theorems that will be of utility in the second part.

Chapter 4 defines the Nash map and states the Nash problem.

Chapter 5 offers a proof of the Nash problem in the affirmative for toric varieties.

Chapter 6 then gives a counterexample in dimension 3, showing that the Nash map is not always bijective.

Notes to the reader

A conscious effort has been made to substantiate every claim with either a proof or a reference to something at least resembling a proof. For the most part, the standard reference for general algebraic geometry has been Grothendieck's [EGA]. Almost all of the theory discussed is documented exclusively in academic papers, and has not seen much action in actual books yet. The bibliography should serve as a robust guide to the modern literature on the Nash problem, although it is by no means exhaustive.

Many of the proofs given are improvements or elaborations on existing proofs in the relevant literature. Emphasis has been put on clarity of exposition through the liberal use of commutative diagrams. It is hoped that this will give the reader a better feeling for how the theory behaves.

Prerequisites

A decent knowledge about schemes and general algebraic geometry is assumed. Chapter 5 also assumes knowledge of the theory of toric varieties.

Assumptions

A ring is always assumed to be commutative and possess a multiplicative identity.

Part I

The theory of arcs

Chapter 1

Jets and arcs

For our purposes the notion of jets is just a stepping stone towards the greater theory of arcs. The next chapter will build the foundational theory of jet schemes without reference to arcs and arc spaces. The purpose of this short chapter is to impart on the reader a vague idea of what an arc is, to keep in mind throughout the next chapter. It should be understood that this is just a brief overview.

> *Assumptions.* Unless otherwise stated, k is an algebraically closed field of arbitrary characteristic.

1.1 Jets

It is informative to define what a jet is before beginning the discussion:

Definition 1.1. Let X be a scheme over k, and let K be a field extension of k. For $m \in \mathbb{N}_0$, a morphism of schemes of the form

$$\operatorname{Spec} \mathrm{K}[t]/(t^{m+1}) \to \mathrm{X}$$

is called an *m-jet on* X.

If we don't care about the value of m for a given m-jet, we just call it a *jet.*

Given a field extension $\mathrm{K} \supseteq k$ and $m \in \mathbb{N}_0$, what does the scheme $\mathrm{Z} := \operatorname{Spec} \mathrm{K}[t]/(t^{m+1})$ look like? Topologically we have a single-point space Spec(Z) containing the only prime ideal (t) of $\mathrm{K}[t]/(t^{m+1})$. In this sense Z is not much more interesting than Spec K, which also contains only one point. However the structure sheaf $\mathcal{O}_{\mathrm{Z}}$ of Z carries a lot more information than that of Spec K, so we can think of the point of Z as "thicker" than the point of Spec K.

Note that $\mathcal{O}_{\mathrm{Z},(t)} = \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}) = \mathcal{O}_{\mathrm{Z}}(\mathrm{Z}) \cong \mathrm{K}[t]/(t^{m+1})$ (cf. [Har77, Proposition II.2.2(c)]). Hence Z supports not only the constant functions $a_0 \in \mathrm{K}$ (which make up the entire structure sheaf $\mathcal{O}_{\operatorname{Spec} \mathrm{K}} \cong \mathrm{K}$), but also polynomial functions up to degree m.

Example 1.2. Consider the situation in Definition 1.1 and suppose that $m = 0$. Then $K[t]/(t^{m+1}) = K[t]/(t) \cong K$, so by the functoriality of Spec we have that $\operatorname{Spec} K[t]/(t) \cong \operatorname{Spec} K$. Hence the 0-jets of X are precisely the morphisms $\operatorname{Spec} K \to X$, i.e. the K-valued points of X. Because X is of finite type over k, we note that if $K = k$ then the 0-jets correspond precisely to the closed points of X.

Take again the situation in Definition 1.1. The 1-jets of X are morphisms $\operatorname{Spec} K[t]/(t^2) \to X$. It turns out that if $K = k$, then morphisms of this form can be regarded as elements of a "tangent bundle" of X. To show this, we start by defining the (Zariski) tangent space of a given scheme.

Definition 1.3. Let X be a scheme over k with structure sheaf $\mathcal{O}_X$. For every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ at x has a unique maximal ideal $\mathfrak{m}_x$. Therefore we may define the *residue field of x on* X, which is the field $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Definition 1.4. Let X be a scheme over k and consider a point $x \in X$. The *Zariski tangent space of* X *at* x, denoted by $T_{X,x}$, is the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Explicitly, we mean that

$$T_{X,x} := (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee = \operatorname{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Proposition 1.5. *Let* X *be a scheme over k and let* K *be a field extension of k. Then to give a morphism* $\operatorname{Spec} k[t]/(t^2) \to X$ *is equivalent to giving a k-valued point x of* X *together with an element of* $T_{X,x}$.

Proof. ($\Longrightarrow$) Let $\alpha : \operatorname{Spec} k[t]/(t^2) \to X$ be a morphism of schemes (a 1-jet). This includes the corresponding morphism of sheaves

$$\alpha^\sharp : \mathcal{O}_X \to \alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)}).$$

We have that $\operatorname{Spec} k[t]/(t^2) = \{(t)\}$, so set $x := \alpha((t)) \in X$. Since (t) is obviously a k-valued point of $\operatorname{Spec} k[t]/(t^2)$, the composition $\operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2) \to X$ has image x and therefore x is a k-valued point.

Now observe that the ring homomorphism α^* corresponding to α is such that

$$\alpha^* = \alpha^\sharp_{(t)} : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} k[t]/(t^2),(t)} \cong k[t]/(t^2),$$

where the isomorphism on the right hand side comes from the fact that

$$\mathcal{O}_{\operatorname{Spec} k[t]/(t^2),(t)} = \Gamma(\operatorname{Spec} k[t]/(t^2), \mathcal{O}_{\operatorname{Spec} k[t]/(t^2)}) \cong k[t]/(t^2).$$

Our goal here is to construct a k-vector space homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. Observe that we have a k-vector space isomorphism $(t)/(t^2) \cong k$ given by $at \mapsto a$ for $a \in k$. Now let us examine what the image of α^* looks like.

Let $\mathfrak{m}_x$ denote the maximal ideal of $k[t]/(t^2)$. Since α^* is local, it follows that $\alpha^*(\mathfrak{m}_x) \subseteq (t)$, where we note that (t) is the maximal ideal of $k[t]/(t^2)$. Moreover, we have that

$$\alpha^*(\mathfrak{m}_x^2) = \{\alpha^*(b^2) \mid b \in \mathfrak{m}_x\} = \{\alpha^*(b)^2 \mid b \in \mathfrak{m}_x\} = (t)^2 = (t^2) = 0.$$

Hence we obtain a k-vector space morphism

$$\alpha^* : \mathfrak{m}_x/\mathfrak{m}_x^2 \to (t)/(t^2) \cong k,$$

which is an element of the Zariski tangent space $\mathrm{T}_{\mathrm{X},x}$ of X at x.

($\Longleftarrow$) Suppose conversely that we are given a k-valued point $x \in \mathrm{X}$ and a k-vector space morphism $f : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. In a natural way we shall construct a morphism of schemes $\alpha : \operatorname{Spec} k[t]/(t^2) \to \mathrm{X}$.

The obvious thing to do is define the topological morphism by $\alpha((t)) := x$. The morphism of sheaves $\alpha^\sharp$ is not so hard either. For any open $\mathrm{U} \subseteq \mathrm{X}$, we do the following: If $x \notin \mathrm{U}$, then note that $\alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})(\mathrm{U}) = \{0\}$, so we set

$$\alpha^\sharp(\mathrm{U}) : \mathcal{O}_\mathrm{X}(\mathrm{U}) \to \alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})(\mathrm{U})$$

to be 0 (what else?). If $x \in \mathrm{U}$, then $\alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})(\mathrm{U}) = k[t]/(t^2)$. Now take the decomposition $\mathcal{O}_{\mathrm{X},x} = k \oplus \mathfrak{m}_x$, which is premissible since x is a k-valued point. Define a ring homomorphism

$$\begin{aligned} \varphi : k \oplus \mathfrak{m}_x &\to k[t]/(t^2) \\ (a,b) &\mapsto a + f(b), \end{aligned}$$

and then set $\alpha^\sharp : \mathcal{O}_\mathrm{X}(\mathrm{U}) \to \alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})(\mathrm{U})$ to be the composition

$$\alpha^\sharp : \mathcal{O}_\mathrm{X}(\mathrm{U}) \to \mathcal{O}_{\mathrm{X},x} = k \oplus \mathfrak{m}_{\mathrm{X},x} \xrightarrow{\varphi} k[t]/(t^2) = \alpha_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})(\mathrm{U}).$$

This completes the proof. □

Roughly speaking, our philosophy is then to take m-jets for $m \gg 1$, thereby thickening the point in $\operatorname{Spec} \mathrm{K}[t]/(t^{m+1})$ and encoding more and more geometric data into the morphism $\operatorname{Spec} \mathrm{K}[t]/(t^{m+1}) \to \mathrm{X}$. This procedure leads us to the notion of arcs.

1.2 Arcs

Let K be a field extension of k, and observe that

$$\mathrm{K}[[t]] = \varprojlim_m \mathrm{K}[t]/(t^{m+1}).$$

This motivates the following definition of an arc:

Definition 1.6. Let X be a scheme over k, and let K be a field extension of k. A morphism of schemes the form

$$\operatorname{Spec} K[[t]] \to X$$

is called an *arc on* X.

The prime spectrum of $K[[t]]$ differs from the prime spectrum of $K[t]/(t^{m+1})$ in that it has two prime ideals, namely (t) *and* (0). Topologically, the ideal (t) is a closed point of $\operatorname{Spec} K[[t]]$ and the zero ideal (0) is open. The zero ideal is also dense, therefore it is the unique generic point of $\operatorname{Spec} K[[t]]$.

Notation 1.7. The generic point (0) of $\operatorname{Spec} K[[t]]$ is denoted by η, and the closed point (t) is denoted by 0. Accordingly, the unique point (t) of $\operatorname{Spec} K[t]/(t^{m+1})$ is denoted by 0 as well.

The difference in topology (as well as the richer scheme structure) of $\operatorname{Spec} K[[t]]$ makes arcs useful little objects with which to probe a scheme. Figure 1.1 offers one way of looking at this: The middle dot depicts the closed point 0, while the ocean of squiggles surrounding it depicts the generic point η. Informally, you could think of arcs as pastings of such objects onto another scheme.

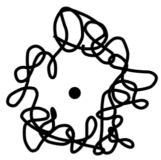


Figure 1.1: Mumford-style interpretation of $\operatorname{Spec} K[[t]]$.

Definition 1.8. The *center* of an arc is the point $\alpha(0)$. For example, one can speak of "an arc with center (resp. centered on) $x \in X$". Equivalently, an arc *through* $x \in X$ means an arc with center $x \in X$.

Although standalone arcs (resp. m-jets) might have something useful to say about the scheme they are hitting, the real power is harnessed by looking at the set of *all* arcs (resp. m-jets), known as *arc spaces* (resp. *m-th jet schemes*). This exercise is also more practical when it comes to building an intuition for these objects. With that in mind, we conclude this chapter and start to build the real theory behind arcs and jets.

Chapter 2

Jet schemes

In this section we shall construct—for any scheme X of finite type over an algebraically closed field—a scheme X_m for each $m \in \mathbb{N}_0$ in which our m-jets can reside. The scheme X_m is known as the *m-th jet scheme of* X.

The construction is not as down-to-earth as just taking the set of all m-jets, but this works to our advantage. Using the flexible notion of *functor of points*, m-jets are associated to points of X_m using morphisms of schemes. By establishing the existence of X_m, a valuable geometric structure is extracted from the m-jets.

Natural morphisms are then constructed between the respective jet schemes in preparation for the next chapter in which the analogous arc space is defined. The arc space will be obtained following the intuition offered in §1.2, namely by taking limits. As the discussion progresses, things smooth out and the theory actually becomes a lot easier to work with.

The means by which the existence of the jet schemes is shown is rather conventional in algebraic geometry, but we will require some ideas from category theory that are worth some attention first. This is where we begin.

> *Assumptions.* Unless otherwise stated, k is an algebraically closed field of arbitrary characteristic.

2.1 Representable functors and the Yoneda lemma

The wealth of information encoded within schemes necessitates that we treat them with some care. For example the underlying topological space reveals deceptively little about a scheme's geometric structure, and it is therefore wholly insufficient to describe it only in this way. This apparent disadvantage of having to account for a lot of mathematical data motivates what turns out to be an extremely elegant and powerful description: the *functor of points*. The strategy—due to Grothendieck—is to describe a scheme by its "interaction" with *all other* schemes.

Although used extensively in algebraic geometry, the functor of points idea is purely category-theoretic and this is language we shall proceed with. See also [Mum99, §II.6], which gives an elegant introduction to its applications.

Definition 2.1. A category $\mathcal{C}$ is said to be *locally small* if for every pair of objects $X, Y \in \mathcal{C}$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of all morphisms $X \to Y$ is a set.

Definition 2.2. Let $\mathcal{C}$ be a locally small category and consider an object $X \in \mathcal{C}$. Define a contravariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{S}\mathrm{et}$ as follows:

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et},$$

carrying

(a) objects $Y \in \mathcal{C}$ to objects $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) \in \mathcal{S}\mathrm{et}$, and

(b) morphisms $f : Y \to Z$ in $\mathcal{C}$ to morphisms

$$\begin{aligned} h_X(f) = \mathrm{Hom}_{\mathcal{C}}(f, X) : h_X(Z) &\to h_X(Y) \\ g &\mapsto g \circ f \end{aligned}$$

in $\mathcal{S}\mathrm{et}$.

We call h_X the *functor of points* of the object X in $\mathcal{C}$.

Definition 2.3. Consider a locally small category $\mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is said to be *representable* if there exists an object $X \in \mathcal{C}$ such that

$$F \cong h_X. \tag{2.1}$$

We have the following additional nomenclature:

(a) A *representation* of F is the pair (X, α): an object $X \in \mathcal{C}$ as above, and an isomorphism α as in (2.1).

(b) A *representing object* of F is an object $X \in \mathcal{C}$ as above.

Remark 2.4. In the context of this discussion it is not worth dwelling on the (nevertheless essential) hypotheses above that $\mathcal{C}$ is locally small. With the exception of functor categories, every category mentioned in this text is locally small (cf. Appendix A). For a more detailed treatment of small, locally small, and large categories, see for example [Lei14, §3.2].

Remark 2.5. The isomorphism (2.1) is in the category $[\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]$ of contravariant functors from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$. Let us briefly examine what is going on here.

Fix two arbitrary categories $\mathcal{A}$ and $\mathcal{B}$, and consider the category $[\mathcal{A}, \mathcal{B}]$. Recall that by convention we call a morphism in this category a *natural transformation*. Similarly an isomorphism is called a *natural isomorphism*. We shall write

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$$

to mean a a natural transformation α from a functor F to a functor G (both of which are objects in the category $[\mathcal{A},\mathcal{B}]$). Then α is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A\in\mathcal{A}}$ of morphisms in $\mathcal{B}$ such that, for each morphism $f : A_1 \to A_2$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{tikzcd} F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\ G(A_1) \arrow[r, "G(f)"'] & G(A_2) \end{tikzcd} \tag{2.2}$$

In this case, we say that α satisfies *naturality*. Note that in the case of $[\mathcal{C}^{op},\mathcal{S}\mathrm{et}]$ above, the horizontal arrows in (2.2) are reversed due to contravariance! See [Lei14, §1.3] and [ML98, §I.4] for a more rigorous overview of natural transformations.

This digression is not without reason, as we can now offer the following result which simplifies the problem of finding representations.

Lemma 2.6. *A natural transformation* $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ *is a natural isomorphism if, and only if,* $\alpha_A : F(A) \to G(A)$ *is an isomorphism for all* $A \in \mathcal{A}$.

Proof. Clear from the discussion in Remark 2.4. □

Returning to the situation in Definition 2.3, we can restate the condition (2.1) as

$$F \cong h_X \iff \exists \begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "F"] \arrow[r, bend right, "h_X"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

$$\overset{L2.6}{\iff} \forall\, Y \in \mathcal{C}\ \ \exists\, \alpha_Y : F(X) \xrightarrow{\cong} h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

The last isomorphism is in the category $\mathcal{S}\mathrm{et}$, meaning that it is simply a bijection. So we have distilled a set-theoretic interpretation of the functor of points, and this will make life much easier in the near future. But although we are tantalisingly close to describing the main result here, we have a crucial observation to make first.

Lemma 2.7 (Yoneda). *Let* $\mathcal{C}$ *be a locally small category and* $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ *a functor. There is a bijection*

$$y : \mathrm{Nat}(h_X, F) \xrightarrow{\cong} F(X)$$
$$\alpha \mapsto \alpha_X(\mathrm{Id}_X)$$

for all $X \in \mathcal{C}$.

Proof. Pick up an $X \in \mathcal{C}$. We are going to construct a morphism

$$y^{-1} : F(X) \to \mathrm{Nat}(h_X, F).$$

Choose an arbitrary $x \in F(X)$. We wish to assign to x a natural transformation

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "h_X"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S}et \end{tikzcd}$$

, i.e. $y^{-1}(x) = \alpha$. This involves defining a family of morphisms $(\alpha_Y : h_X(Y) \to F(Y))_{Y \in \mathcal{C}}$ which satisfies naturality. So we require that for each morphism $f : Y_2 \to Y_1$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} h_X(Y_1) \arrow[r, "h_X(f)"] \arrow[d, "\alpha_{Y_1}"'] & h_X(Y_2) \arrow[d, "\alpha_{Y_2}"] \\ F(Y_1) \arrow[r, "F(f)"'] & F(Y_2) \end{tikzcd} \tag{2.3}$$

(Note that for clarity we have reversed the arrows in $\mathcal{C}$ because we are working with contravariant functors, cf. Remark 2.5.)

The assignment we choose is as follows: for $Y \in \mathcal{C}$,

$$\begin{aligned} \alpha_Y : h_X(Y) &\to F(Y) \\ g &\mapsto F(g)(x). \end{aligned}$$

Naturality follows immediately. Pick up a morphism $f : Y_2 \to Y_1$ in $\mathcal{C}$. Then commutativity in (2.3) is verified as follows:

$$\begin{tikzcd} g \arrow[rr, mapsto, "-\circ f", "\mathrm{D2.2}"'] \arrow[dd, mapsto] & & g \circ f \arrow[d, mapsto] \\ & \begin{array}{ccc} h_X(Y_1) & \xrightarrow{h_X(f)} & h_X(Y_2) \\ \downarrow{\scriptstyle \alpha_{Y_1}} & & \downarrow{\scriptstyle \alpha_{Y_2}} \\ F(Y_1) & \xrightarrow[F(f)]{} & F(Y_2) \end{array} & F(g \circ f)(x) \arrow[d, equal] \\ F(g)(x) \arrow[rr, mapsto] & & F(f)F(g)(x) \end{tikzcd}$$

Since F is a functor, we know that $F(f)F(g) = F(g \circ f)$, whence the bottom right equality. Our choice of $x \in F(X)$ was arbitrary, so the morphism $y^{-1} : F(X) \to \mathrm{Nat}(h_X, F)$ is well-defined.

It remains to check that y^{-1} is—as the label suggests—a two-sided inverse of y. For every $x \in F(X)$ we have

$$y \circ y^{-1}(x) = (y^{-1}(x))_X(\mathrm{Id}_X) = F(\mathrm{Id}_X)(x) = \mathrm{Id}_{F(X)}(x).$$

Next, for every $\alpha \in \mathrm{Nat}(h_X, F)$, $Y \in \mathcal{C}$, and $g \in h_X(Y)$ we have

$$(y^{-1} \circ y(\alpha))_Y(g) = (y^{-1}(\alpha_X(\mathrm{Id}_X)))_Y(x) = F(g)(\alpha_X(\mathrm{Id}_X)) = \alpha_Y(g).$$

Here, the last equality can be seen by running $\mathrm{Id}_X \in h_X(X)$ through the following commutative diagram:

$$\begin{tikzcd} h_X(X) \arrow[r, "h_X(g)"] \arrow[d, "\alpha_X"'] & h_X(Y) \arrow[d, "\alpha_Y"] \\ F(X) \arrow[r, "F(g)"'] & F(Y) \end{tikzcd}$$

So y is indeed a bijection.

Our original choice of $X \in \mathcal{C}$ was arbitrary, whence the result. □

The following corollary is the main result of this section. It crystalises this idea of constructing objects in a category using the functor of points.

Corollary 2.8. *Let $\mathcal{C}$ be a locally small category and let $X, X' \in \mathcal{C}$. If $h_X \cong h_{X'}$, then there exists a unique isomorphism $X \cong X'$.*

Proof. Suppose that α is the natural isomorphism between h_X and $h_{X'}$. Then the main thing to observe here is that we get a morphism

$$y(\alpha) \stackrel{\text{L2.7}}{=} \alpha_X(\mathrm{Id}_X) \in h_{X'}(X),$$

and that α_X^{-1} is a two-sided inverse:

$$\begin{aligned} \alpha_X^{-1}(\alpha_X(\mathrm{Id}_X)) &= \mathrm{Id}_X \\ \alpha_X(\alpha_X^{-1}(\mathrm{Id}_X)) &= \mathrm{Id}_{X'} . \end{aligned}$$

So $y(\alpha)$ is the desired isomorphism. Uniqueness follows from the bijectivity of y and the uniqueness of the isomorphism $h_X \cong h_{X'}$. □

As a result, we say that representing objects are *unique up to canonical isomorphism.*

The following result will make life easier too:

Lemma 2.9. *For any ring R, a scheme over R is determined by the restriction of its functor of points to the affine schemes over R.*

Proof. By glueing. See [EH00, Proposition VI-2]. □

Now armed with the above arsenal of category theory, we are ready to describe jet schemes in their totality. We will use the results stated above freely and without explicit reference.

2.2 Construction of jet schemes

As has already been noted, the focus here will be on schemes of finite type over k. The reason for the finite type condition is purely pratical. Except for in the final chapter, we will have little utility for a more general theory, and to present one would require quite a lot more work. See also Remark 3.8.

We have already observed in Example 1.2 that the 0-jets on $X \in \mathcal{S}ch_{ft}/k$ correspond precisely to the K-valued points of X. We can then suppose that the K-valued points of X_0 correspond precisely to the 0-jets on X, and similarly for other values of $m \in \mathbb{N}_0$. This leads us to the following definition:

Definition 2.10. Let X be a scheme of finite type over k and let $m \in \mathbb{N}_0$. Consider the contravariant functor

$$\begin{aligned} F_m^X : \mathcal{S}ch/k &\to \mathcal{S}et \\ Z &\mapsto \mathrm{Hom}_{\mathcal{S}ch/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X). \end{aligned}$$

The *m-th jet scheme of* X is the the representing object $X_m \in \mathcal{S}ch/k$ of F_m^X. Explicitly, we require X_m such that

$$\mathrm{Hom}_{\mathcal{S}ch/k}(Z, X_m) \cong \mathrm{Hom}_{\mathcal{S}ch/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X) \tag{2.4}$$

for all $Z \in \mathcal{S}ch/k$.

It is not immediately obvious that such an $X_m \in \mathcal{S}ch/k$ exists. However if it does, then we know that it is unique up to canonical isomorphism (cf. Corollary 2.8). As is common for such results, we shall first prove the existence of X_m in the simpler case where X is an affine scheme. This will lead to the general result for all schemes X through glueing.

Proposition 2.11. *Let* X *be an affine scheme of finite type over k. The m-th jet scheme* X_m *exists for all* $m \in \mathbb{N}_0$.

Proof. Fix $m \in \mathbb{N}_0$. As X is affine, we have $X = \operatorname{Spec} R$ for some suitable ring R. Moreover, X is of finite type, meaning that R is a finitely generated k-algebra. Hence we can express R as

$$R = k[x_1, \dots, x_n]/I,$$

for suitable $I = (f_1, \dots, f_r)$ and $n, r \in \mathbb{N}$.

Fix an arbitrary $m \in \mathbb{N}_0$. By Lemma 2.9, it suffices to consider an affine scheme $Z = \operatorname{Spec} A$, where A is a k-algebra.

We wish to show that

$$\mathrm{Hom}_{\mathcal{S}ch/k}(\operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X) \cong \mathrm{Hom}_{\mathcal{S}ch/k}(\operatorname{Spec} A, X_m)$$

for a suitable $X_m \in \mathcal{S}\mathrm{ch}/k$. First note that

$$\begin{aligned}\operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) &= \operatorname{Spec}\left(A \otimes_k k[t]/(t^{m+1})\right) \\ &\cong \operatorname{Spec} A[t]/(t^{m+1}).\end{aligned} \tag{2.5}$$

Using this fact, we have that

$$\begin{aligned}&\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X) \\ &\cong \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), \operatorname{Spec} R) \\ &\cong \operatorname{Hom}_{k-\mathcal{A}\mathrm{lg}}(R, A[t]/(t^{m+1})) \\ &= \operatorname{Hom}_{k-\mathcal{A}\mathrm{lg}}(k[x_1, \ldots, x_n]/I, A[t]/(t^{m+1})) \\ &\cong \left\{\varphi \in \operatorname{Hom}_{k-\mathcal{A}\mathrm{lg}}(k[x_1, \ldots, x_n], A[t]/(t^{m+1})) \mid \varphi(f_\ell) = 0 \quad (\ell = 1, \ldots, r)\right\}.\end{aligned} \tag{2.6}$$

Pick up a φ as in (2.6). Write

$$\varphi(x_i) = a_{i0} + a_{i1}t + \cdots + a_{im}t^m \in A[t]/(t^{m+1}) \tag{2.7}$$

for $i = 1, \ldots, n$ and suitable coefficients $\{a_{ij}\}_{i=1,j=0}^{i=n,j=m}$ in A (we will drop the indices for clarity). These coefficients then then uniquely determine φ.

Since the f_ℓ are polynomials over k in the variables $x_1, \ldots, x_n$, (2.7) allows us to express each $\varphi(f_\ell)$ entirely in terms of the a_{ij}. This works because φ is a k-algebra morphism and is hence linear in k. The coefficients of $\varphi(f_\ell)$ will then be linear combinations of the a_{ij}, so we can think of them as polynomials over k in $n(m+1)$ variables. Explicitly this means writing

$$\varphi(f_\ell) = F_{\ell 0}(\{a_{ij}\}) + F_{\ell 1}(\{a_{ij}\})t + \cdots + F_{\ell m}(\{a_{ij}\})t^m$$

for $\ell = 1, \ldots, r$ and suitable polynomials $\{F_{\ell s}\}_{\ell=1,s=0}^{\ell=r,s=m}$.

This polynomial interpretation of the coefficients of $\varphi(f_\ell)$ makes it clear that the $F_{\ell s}$ are independent of the choice of φ. To see this, suppose that we pick some other φ' from the set in (2.6) and set the a_{ij} such that

$$\varphi'(x_i) = a_{i0} + a_{i1}t + \cdots + a_{im}t^m,$$

just as we did in (2.7). Then we construct the $F_{\ell s}$ in exactly the same way as above, using the k-linearity of φ'. Indeed, we have shown that the $F_{\ell s}$ depend only on the f_ℓ.

Now note the condition that $\varphi(f_\ell) = 0$ for $\ell = 1, \ldots, r$. A polynomial is identically zero if, and only if, all of its coefficients are zero. Hence

$$F_{\ell s}(\{a_{ij}\}) = 0 \tag{2.8}$$

for $\ell = 1, \ldots, r$ and $s = 0, \ldots, m$.

So what have we done here? We have taken each element φ of the set (2.6) and expressed it uniquely as collection of elements $\{a_{ij}\}$ in A. Moreover, we have said that the condition

$$\varphi(f_\ell) = 0$$

is equivalent to the condition (2.8).

The next step is to reinterpret $\varphi : k[x_1, \dots, x_n] \to A[t]/(t^{m+1})$ as a k-algebra morphism $k[\{x_{ij}\}_{i=1,j=0}^{i=n,j=m}] \to A$ instead. This is easy, for we have a natural bijection

$$\mathrm{Hom}_{k-\mathcal{A}\mathrm{lg}}(k[x_1, \dots, x_n], A[t]/(t^{m+1})) \cong \mathrm{Hom}_{k-\mathcal{A}\mathrm{lg}}(k[\{x_{ij}\}], A)$$

which sends φ to

$$\begin{aligned} \varphi : k[\{x_{ij}\}] &\to A \\ x_{ij} &\mapsto a_{ij}. \end{aligned}$$

(Again we are dropping indices for clarity.) With this in mind, we continue (2.6) as follows:

$$\begin{aligned} \cdots &\cong \{\varphi \in \mathrm{Hom}_k(k[\{x_{ij}\}], A) \mid a_{ij} := \varphi(x_{ij}),\ F_{\ell s}(\{a_{ij}\}) = 0\} \\ &= \{\varphi \in \mathrm{Hom}_k(k[\{x_{ij}\}], A) \mid F_{\ell s}(\{\varphi(x_{ij})\}) = 0\} \\ &= \{\varphi \in \mathrm{Hom}_k(k[\{x_{ij}\}], A) \mid \varphi(F_{\ell s}(\{x_{ij}\})) = 0\} \\ &\cong \mathrm{Hom}_{k-\mathcal{A}\mathrm{lg}}(\underbrace{k[\{x_{ij}\}]/(\{F_{\ell s}(\{x_{ij}\})\})}_{=:R_m}, A), \end{aligned}$$

for $i \in \{1, \dots, n\}$, $j, s \in \{0, \dots, m\}$, $\ell \in \{1, \dots, r\}$, and setting $X_m := \operatorname{Spec} R_m$,

$$\cong \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X_m).$$

As a courtesy to the reader, here is the fully indexed definition of R_m:

$$R_m = k[\{x_{ij}\}_{i=1,j=0}^{i=n,j=m}]/(\{F_{\ell s}(\{x_{ij}\}_{i=1,j=0}^{i=n,j=m})\}_{\ell=1,s=0}^{\ell=r,s=m}).$$

Critically, we noted that the $F_{\ell s}$ depend only on the f_ℓ. Hence—by some definition chasing—the $F_{\ell s}$ depend only on X. In particular, our choice of X_m is independent of our choice of Z, whence the result. □

Corollary 2.12. *All jet schemes of affine schemes (of finite type over k) are affine.*

Remark 2.13. By Lemma 2.9 and (2.5), we can restate Definition 2.10 as follows: The m-th jet scheme of a scheme X of finite type over k is the scheme $X_m \in \mathcal{S}\mathrm{ch}/k$ such that

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X_m) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), X) \tag{2.9}$$

for all $A \in k{-}\mathcal{A}\mathrm{lg}$.

This definition ends up being more intuitive and useful for most of our purposes. Indeed it is the main definition used by many authors on the subject, for example [DL99], [Vey06], and [EM09].

Remark 2.14. Noting that $A[t]/(t) \cong A$ for all $A \in k\text{-}\mathcal{A}\mathrm{lg}$, it becomes obvious that

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X_0) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t), X) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X)$$

and hence $X_0 \cong X$. This fits with our description at the beginning of this section, rather unsurprisingly. Because this is a unique canonical isomorphism, we shall usually just write $X_0 = X$.

2.3 Examples of jet schemes

Having undertaken the somewhat laborious task of proving existence in the affine case, let's take a break and admire some examples.

Example 2.15 (Affine space). Let $X := \mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$ for $n \in \mathbb{N}_0$. Then $X_m = \operatorname{Spec} k[\{x_{ij}\}_{i=1,j=0}^{i=n,j=m}] \cong \mathbb{A}_k^{(m+1)n}$ for all $m \in \mathbb{N}_0$. This is clear from the definition of R_m given in the proof of Proposition 2.11 for $I = (0)$.

The above example essentially comes for free, but in general our strategy of computation is as follows: As we noted at the end of the proof of Proposition 2.11, the polynomials $F_{\ell s}$ depend only on the choice of X. Unwinding the construction given above, we then get the general formula

$$X_m = \{\varphi = (\varphi_1, \dots, \varphi_n) \in (k[t]/(t^{m+1}))^n \mid f_\ell(\varphi) = 0 \mod t^{m+1} \quad (\ell = 1, \dots, r)\}$$

for $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$, $m \in \mathbb{N}_0$. This is guaranteed to make more sense after the next example.

Notation 2.16. Note that in the formula given above, we are abusing notation. It is implied that we are taking the prime spectrum of the ring described, and therefore we omit the Spec. Moreover, we will allow ourselves to write something like

$$X = \{x^3 - y = 0\} = \{(x, y) \in k^2 \mid x^3 - y = 0\} = \{x + yt \in k[t]/(t^{m+1}) \mid x^3 - y = 0\},$$

by which we mean the prime spectrum of the far right ring. In particular, these sets correspond to the closed (or k-valued) points of X, so we get to see the traditional variety interpretation a bit better by writing things like this.

However, this abuse of notation will only really be done in examples.

Example 2.17 (Parabola). Let $X := \{x^2 - y = 0\} = \operatorname{Spec} k[x, y]/(x^2 - y)$. Following the explanation given above, we have:

(a) $X_0 = \{(a_0, b_0) \in k^2 \mid a_0^2 - b_0 = 0\} = X$. This is trivial, but helps with the next computations.

(b) We have

$$X_1 = \Big\{(a_0 + a_1 t, b_0 + b_1 t) \in (k[t]/(t^2))^2 \;\Big|\; (a_0 + a_1 t)^2 - (b_0 + b_1 t) = 0 \mod t^2\Big\},$$

so let's write this out and see what equations we get: $(a_0+a_1t)^2-b_0-b_1t = a_0^2+2a_0a_1t+\cancel{a_1^2t^2}-b_0-b_1t = a_0^2+2a_0a_1t-b_0-b_1t$. Hence the equation becomes $a_0^2+2a_0a_1t-b_0-b_1t = 0$. In fact the variable t is going to vanish, as is clear by the following rearrangement:

$$\underbrace{a_0^2 - b_0}_{=0} + t\underbrace{(2a_0a_1 - b_1)}_{=0} = 0.$$

In particular, we fall back to the simple fact that a polynomial is identically zero if, and only if, its coefficients are all zero. This gives us two equations:

$$a_0^2 - b_0 = 0, \tag{2.10}$$

(note that this is the condition we had for X_0!), and

$$2a_0a_1 - b_1 = 0. \tag{2.11}$$

Combining (2.10) and (2.11), we can continue our computation of X_1 above:

$$\begin{aligned} &\vdots \\ &= \left\{(a_0 + a_1 t, b_0 + b_1 t) \in (k[t]/(t^2))^2 \;\middle|\; \begin{matrix} a_0^2 - b_0 = 0 \\ 2a_0a_1 - b_1 = 0 \end{matrix} \right\} \\ &= \left\{(a_0, a_1, b_0, b_1) \in k^4 \;\middle|\; \begin{matrix} a_0^2 - b_0 = 0 \\ 2a_0a_1 - b_1 = 0 \end{matrix} \right\}. \end{aligned}$$

(c) One more. Look at

$$X_2 = \Big\{(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \in (k[t]/(t^3))^2 \;\Big|\; (a_0 + a_1 t + a_2 t^2)^2 - (b_0 + b_1 t + b_2 t^2) = 0 \mod t^3\Big\}.$$

Expanding the left hand side of the equation just like before, we get $(a_0+a_1t+a_2t^2)^2-b_0-b_1t-b_2t^2 = a_0^2+2a_0a_1t+2a_0a_2t^2+a_1^2t^2+\cancel{2a_1a_2t^3}+\cancel{a_2^2t^4}-b_0-b_1t-b_2t^2 = a_0^2+2a_0a_1t+2a_0a_2t^2+a_1^2t^2-b_0-b_1t-b_2t^2$. Rearrange:

$$\underbrace{a_0^2-b_0}_{(2.10)}+t(\underbrace{2a_0a_1-b_1t}_{(2.11)})+t^2(\underbrace{2a_0a_2+a_1^2-b_2}_{=0})=0.$$

The first two coefficients correspond to the conditions for X_1 that we computed above (see the pattern?), and we are left with one new equation

$$2a_0a_2+a_1^2-b_2=0. \tag{2.12}$$

Putting this together, we then have X_3:

$$\begin{aligned}
&\vdots\\
&=\left\{\begin{array}{c}(a_0+a_1t+a_2t^2,\\ b_0+b_1t+b_2t^2)\end{array}\in (k[t]/(t^3))^2 \;\middle|\; \begin{array}{l}a_0^2-b_0=0\\ 2a_0a_1-b_1=0\\ 2a_0a_2+a_1^2-b_2=0\end{array}\right\}\\
&=\left\{(a_0,a_1,a_2,b_0,b_1,b_2)\in k^6 \;\middle|\; \begin{array}{l}a_0^2-b_0=0\\ 2a_0a_1-b_1=0\\ 2a_0a_2+a_1^2-b_2=0\end{array}\right\}.
\end{aligned}$$

The next example is given without explicit computation.

Example 2.18 (Cusp). Let $X := \{x^3-y^2=0\} = \operatorname{Spec} k[x,y]/(x^3-y^2)$.

(a) $X_0 = \{(a_0,b_0)\in k^2 \mid a_0^3-b_0^2=0\} = \operatorname{Spec} k[x,y]/(x^3-y^2) = X$. This agrees with what was said in Remark 2.14.

(b)

$$\begin{aligned}
X_1 &= \Big\{(a_0+a_1t, b_0+b_1t)\in (k[t]/(t^2))^2 \;\Big|\\
&\qquad\qquad (a_0+a_1t)^3-(b_0+b_1t)^2=0 \mod t^2\Big\}\\
&=\left\{(a_0+a_1t, b_0+b_1t)\in (k[t]/(t^2))^2 \;\middle|\; \begin{array}{l}a_0^3-b_0^2=0\\ 3a_0^2a_1-2b_0b_1=0\end{array}\right\}\\
&=\left\{(a_0,a_1,b_0,b_1)\in k^4 \;\middle|\; \begin{array}{l}a_0^3-b_0^2=0\\ 3a_0^2a_1-2b_0b_1=0\end{array}\right\}
\end{aligned}$$

(c)

$$\begin{aligned}
X_2 &= \Big\{(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \in (k[t]/(t^3))^2 \;\Big|\; \\
&\qquad (a_0 + a_1 t + a_2 t^2)^3 - (b_0 + b_1 t + b_2 t^2)^2 = 0 \mod t^3\Big\} \\
&= \left\{ \begin{matrix} (a_0 + a_1 t + a_2 t^2, \\ b_0 + b_1 t + b_2 t^2) \end{matrix} \in (k[t]/(t^3))^2 \;\middle|\; \begin{matrix} a_0^3 - b_0^2 = 0 \\ 3a_0^2 a_1 - 2b_0 b_1 = 0 \\ 3a_0 a_1^2 + 3a_0^2 a_2 - b_1^2 - 2b_0 b_2 = 0 \end{matrix} \right\} \\
&= \left\{ (a_0, a_1, a_2, b_0, b_1, b_2) \in k^6 \;\middle|\; \begin{matrix} a_0^3 - b_0^2 = 0 \\ 3a_0^2 a_1 - 2b_0 b_1 = 0 \\ 3a_0 a_1^2 + 3a_0^2 a_2 - b_1^2 - 2b_0 b_2 = 0 \end{matrix} \right\}
\end{aligned}$$

Although we haven't said much about these examples yet, we shall revisit and expand on them as the discussion progresses. But for now, a return to the general theory.

2.4 Jets from points, and vice versa

From the above discussion we can associate to each point in a jet scheme X_m to an m-jet on X in the following way: For each $x \in X_m$, there exists a canonical morphism

$$i_x : \operatorname{Spec} k(x) \to X_m$$

with image x (cf. [Mum99, p. 116]). By (2.9) it corresponds uniquely to a morphism

$$\alpha_x : \operatorname{Spec} k(x)[t]/(t^{m+1}) \to X.$$

Note that the residue field $k(x)$ is a field extension of k. This follows from the fact that X_m is a scheme over k, and therefore the diagonal morphism of the following commutative diagram corresponds to a *local and hence injective* ring homomorphism $k \hookrightarrow k(x)$:

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r, "i_x"] \arrow[dr] & X_m \arrow[d] \\
& \operatorname{Spec} k
\end{tikzcd}$$

So α_x is an m-jet on X.

Conversely, any m-jet $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \to X$ corresponds to a morphism

$$\operatorname{Spec} K \to X_m$$

whose image is a point on X_m. This is the inverse operation to the above mapping $x \mapsto \alpha_x$. However, there are of course different m-jets that correspond to the same point on X_m.

From now on we shall usually make little distinction between a point on X_m and an m-jet on X. In particular, we shall refer to a point and its corresponding jet by the same symbol (typically α). In the above discussion, this means that $x \in X_m$ and $\alpha_x : \operatorname{Spec} k(x)[t]/(t^{m+1}) \to X$ are both relabelled α. We then say that $\alpha \in X_m$ is an m-jet on X.

2.5 Truncation morphisms

Having proved the existence of jet schemes in the affine case, it is quite easy to generalise the result for all schemes of finite type over k. The proof is a straightforward glueing procedure, but first we must describe the device with which we shall glue. Namely, we are looking to construct a natural morphism $X_m \to X$.

Let A be a k-algebra. Consider the natural projection given by truncation of polynomials

$$\begin{aligned} p^* : A[t]/(t^{m+1}) &\to A[t]/(t^{n+1}) \\ \sum_{i=0}^{m} \lambda_i t^i &\mapsto \sum_{i=0}^{n} \lambda_i t^i \end{aligned} \tag{2.13}$$

for $m, n \in \mathbb{N}_0$, $m \geq n$. It is clear that this is a ring homomorphism, and hence corresponds to a morphism of schemes

$$p : \operatorname{Spec} A[t]/(t^{n+1}) \to \operatorname{Spec} A[t]/(t^{m+1}).$$

Now suppose that X_m exists for some scheme X of finite type over k (we have already shown this to be true if X is affine). Given some m-jet $\alpha \in X_m$, i.e. a morphism

$$\alpha : \operatorname{Spec} A[t]/(t^{m+1}) \to X,$$

there is an induced morphism given by composition of α with the morphism p

$$\begin{tikzcd} \operatorname{Spec} A[t]/(t^{n+1}) \arrow[r, "p"] \arrow[dr, dotted] & \operatorname{Spec} A[t]/(t^{m+1}) \arrow[d, "\alpha"] \\ & X \end{tikzcd}$$

and it is an n-jet on X. This procedure defines what we call the *truncation morphisms*, given by

$$\begin{aligned} \psi_{m,n} : X_m &\to X_n \\ \alpha &\mapsto \alpha \circ p \end{aligned} \tag{2.14}$$

for all $m, n \in \mathbb{N}_0$, $m \geq n$. Recalling that the 0-jet scheme is X (cf. Remark 2.14), we denote the morphism $\psi_{m,0}$ by π_m for all $m \in \mathbb{N}_0$. This gives us an A-valued point $\pi_m(\alpha)$ for every such α. If it is ambiguous which jet scheme the truncation morphisms correspond to, then we write $\psi_{X,m,n}$ and $\pi_{X,m}$ respectively.

If $A = K$ for some field extension $K \supseteq k$, then $\pi_m(\alpha) = \alpha(0)$. Remember here that 0 denotes the unique point (t) of $\operatorname{Spec} K[t]/(t^{m+1})$. In particular, given any subset $Z \subseteq X$, the K-valued points of the preimage $\pi_m^{-1}(Z)$ are precisely the m-jets of X with image in Z.

This is all we need. The following lemma is our glue, and the subsequent proposition is our desired existence result.

Lemma 2.19. *Let X be a scheme of finite type over k, and consider an open subscheme $U \subseteq X$. If X_m exists, then U_m exists and $U_m \cong \pi_m^{-1}(U)$.*

Proof. Let $A \in k\text{-}\mathcal{A}\mathrm{lg}$ be arbitrary. We have the situation

$$\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, U_m) \cong \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), U), \tag{2.15}$$

and we are going to construct a bijection

$$\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, \pi_m^{-1}(U)) \cong \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), U) \tag{2.16}$$

from which it is evident that $U_m \cong \pi_m^{-1}(U)$.

Pick up a morphism $\alpha : \operatorname{Spec} A[t]/(t^{m+1}) \to X$, which can be regarded as an element of both $\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X_m)$ and $\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), X)$ by virtue of the correspondence (2.15). In order to construct the bijection (2.16), it then suffices to show that

$$\alpha \in \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, \pi_m^{-1}(U)) \iff \alpha \in \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), U).$$

Keeping the same notation as in (2.13), the rest of the proof is indicated by the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, X_m) \arrow[rr, phantom, "\cong"] & & \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), X) \\
\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A, \pi_m^{-1}(U)) \arrow[u, hook] \arrow[rr, "(2.16)"] & & \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(\operatorname{Spec} A[t]/(t^{m+1}), U) \arrow[u, hook] \\
\alpha \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] \arrow[d, Leftrightarrow] & & \alpha \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, Leftrightarrow] \\
\operatorname{Im}(\pi_m(\alpha)) \subseteq U \arrow[r, Leftrightarrow] & \operatorname{Im}(\alpha \circ p) \subseteq U \arrow[r, Leftrightarrow] & \operatorname{Im}(\alpha) \subseteq U
\end{tikzcd}$$

In particular, $\pi_m^{-1}(U)$ is a representing object of the functor F_m^U (cf. Definition 2.10), whence the result. ◻

Proposition 2.20. *Let X be a scheme over k of finite type. The m-th jet scheme X_m exists for all $m \in \mathbb{N}_0$.*

Proof. As X is of finite type, there exists a finite open cover $\{U_i\}_{i=1}^r$ of affine schemes $U_i = \operatorname{Spec} A_i$ where each A_i is a finitely generated k-algebra. Fix an arbitrary $m \in \mathbb{N}_0$. It follows from Proposition 2.11 that the m-th jet scheme $(U_i)_m$ exists for all $i = 1, \ldots, r$.

Observe that by Lemma 2.19,

$$\pi_{U_i,m}^{-1}(U_i \cap U_j) \cong (U_i \cap U_j)_m \cong \pi_{U_j,m}^{-1}(U_i \cap U_j) \tag{2.17}$$

for all $i, j = 1, \ldots, r$. Then define $X_m \in \mathcal{S}ch/k$ to be the scheme obtained by glueing the schemes $(U_i)_m$ along the isomorphisms given in (2.17).

Is this X_m good enough? We must check that there exists a bijection

$$\operatorname{Hom}_{\mathcal{S}ch/k}(Z, X_m) \cong \operatorname{Hom}_{\mathcal{S}ch/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X)$$

for all $Z \in \mathcal{S}ch/k$ (cf. Definition 2.10). Pick up an arbitrary $\alpha \in \operatorname{Hom}_{\mathcal{S}ch/k}(Z, X_m)$. For each $i, j = 1, \ldots, r$, consider the open affine subschemes

$$Z_i := \alpha^{-1}((U_i)_m) \subseteq Z,$$

and the open subschemes

$$Z_{ij} = Z_{ji} := \alpha^{-1}((U_i \cap U_j)_m) \subseteq Z_i.$$

Clearly we recover Z by glueing the Z_i together along the Z_{ij}. Moreover, α restricted to Z_{ij} is an element of $(U_i \cap U_j)_m$ and so each restriction corresponds to a unique $\alpha_{ij} \in \operatorname{Hom}_{\mathcal{S}ch/k}(Z_{ij} \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), U_i \cap U_j)$. Noting that $Z_i = Z_{ii}$, this also means that α restricted to Z_i corresponds to a unique $\alpha_i \in \operatorname{Hom}_{\mathcal{S}ch/k}(Z_i \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), U_i)$.

By glueing the $Z_i \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})$ along the $Z_{ij} \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})$, we obtain $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})$. Hence the α_i define a unique morphism $\alpha \in \operatorname{Hom}_{\mathcal{S}ch/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X)$ and we have an injection

$$\operatorname{Hom}_{\mathcal{S}ch/k}(Z, X_m) \hookrightarrow \operatorname{Hom}_{\mathcal{S}ch/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Reversing this process follows exactly the same logic and yields the desired bijection. Therefore X_m is the m-th jet scheme of X. $\square$

Remark 2.21. The fact that Lemma 2.19 ensures the existence of U_m for any open $U \subseteq X$ is critical if the above proof of Proposition 2.20 is to work. Observe that in the glueing procedure, we took for granted that $(U_i \cap U_j)_m$ is a well-defined object. What are the consequences if we suppose that the existence of this jet scheme is unknown to us?

If $U_i \cap U_j$ is not affine, then we can not argue that its jet schemes exist. This is a very real threat, for while it is tempting to assume that the intersection of two open affine subschemes will be affine, this is in general false.[1] The proof would then fail.

Fortunately, this was not the case.

Corollary 2.22. *All jet schemes (of schemes of finite type over k) are of finite type over k.*

2.6 Further examples

Example 2.23 (Affine space, continued). Recall from Example 2.15 that for $X := \mathbb{A}^n_k$, we have $X_m = \operatorname{Spec} k[\{x_{ij}\}_{i=1,j=0}^{i=n,j=m}] \cong \mathbb{A}_k^{(m+1)n}$ for each $m \in \mathbb{N}_0$. What do the truncation morphisms $\psi_{m,m'}$ $(m \geq m')$ look like?

Pick up the canonical morphism $p : \operatorname{Spec} A[t]/(t^{m'+1}) \to \operatorname{Spec} A[t]/(t^{m+1})$ induced by polynomial truncation, and the truncation morphism $\psi_{m,m'} : X_m \to X_{m'}$. Any $\alpha \in X_m$ can be regarded as a morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X = \operatorname{Spec} k[x_1, \dots, x_n]$ and hence corresponds to a k-algebra homomorphism

$$\alpha^* : k[x_1, \dots, x_n] \to A[t]/(t^{m+1}).$$

Then note that

$$(\alpha \circ p)^* = p^* \circ \alpha^* : k[x_1, \dots, x_n] \xrightarrow{\alpha^*} A[t]/(t^{m+1}) \xrightarrow{p^*} A[t]/(t^{m'+1}),$$

which corresponds to the morphism of k-schemes

$$\psi_{m,m'}(\alpha) = \alpha \circ p : \operatorname{Spec} A \to X_m.$$

Of course this gives an A-valued point on X_m, but more interesting is the following behaviour:

$$\psi_{m,m'} : \mathbb{A}_k^{(m+1)n} \cong X_m \to X_{m'} \cong \mathbb{A}_k^{(m'+1)n}$$
$$\begin{bmatrix} a_{1,0} & \cdots & a_{1,m'} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,0} & \cdots & a_{n,m'} & \cdots & a_{n,m} \end{bmatrix} \mapsto \begin{bmatrix} a_{1,0} & \cdots & a_{1,m'} \\ \vdots & \ddots & \vdots \\ a_{n,0} & \cdots & a_{n,m'}. \end{bmatrix}$$

This is readily verified by inspection of the corresponding k-algebra homomorphism $(\alpha \circ p)^*$. And by projecting onto only the first $(m'+1)n$ variables, the morphism $\psi_{m,m'}$ earns its title "truncation".

With this in mind, it follows that for any subscheme $Z \subseteq X_{m'}$,

$$\psi_{m,m'}^{-1}(Z) \cong Z \times_{\operatorname{Spec} k} \mathbb{A}_k^{(m-m')n}$$

[1] If X is separated, then an intersection of such affine opens *will* be affine; see [EGA I, Proposition 5.5.6]. Although non-separated schemes are somewhat pathological objects compared to their separated cousins, it turns out that to prove existence of their jet schemes did not need much extra work.

for all $m, m' \in \mathbb{N}_0$, $m \geq m'$. In particular for every k-valued point $x \in \mathrm{X}$,

$$\pi_m^{-1}(\{x\}) \cong \mathbb{A}_k^{mn}.$$

In the next examples, we will work only with closed points. The idea is to give a general picture of the behaviour, which should serve as inspiration for the following chapter where the results are formally proven.

Example 2.24 (Parabola, continued). Let $\mathrm{X} := \{x^2 - y = 0\}$ as in Example 2.17. For any k-valued point $(x,y) \in \mathrm{X}$,

$$\pi_1^{-1}((x,y)) = \{(x, a_1, y, b_1) \in \mathbb{A}_k^4 \mid 2xa_1 - b_1 = 0\},$$

which corresponds to a line in the (a_1, b_1)-plane with equation $2xa_1 - b_1 = 0$. This is precisely the tangent space $\mathrm{T}_{\mathrm{X},(x,y)}$ of X at (x,y) (cf. Proposition 1.5).

Similarly, we see that

$$\pi_2^{-1}((x,y)) = \left\{ (x, a_1, a_2, y, b_1, b_2) \in k^6 \;\middle|\; \begin{array}{l} 2xa_1 - b_1 = 0 \\ 2xa_2 + a_1^2 - b_2 = 0 \end{array} \right\},$$

which represents the hypersurface in (a_1, a_2, b_1, b_2) with equations $2xa_1 - b_1 = 0$ and $2xa_2 + a_1^2 - b_2 = 0$. If $(x,y) = (0,0)$, then we get

$$\begin{aligned} \pi_2^{-1}((0,0)) &= \{(0, a_1, a_2, 0, 0, b_2) \in k^6 \mid a_1^2 - b_2 = 0\} \\ &\cong \{(a_1, a_2, b_2) \in k^3 \mid a_1^2 - b_2 = 0\} \\ &\cong \mathrm{X} \times_{\operatorname{Spec} k} \mathbb{A}_k^1. \end{aligned}$$

Since the parabola X is smooth, the truncation morphisms $\psi_{m,m-1}$ are in fact surjective. Here's an example of how $\psi_{2,1}$ maps $\pi_2^{-1}((x,y))$ to $\pi_1^{-1}((x,y))$:

$$\psi_{2,1} : \pi_2^{-1}((x,y)) \to \pi_1^{-1}((x,y))$$

$$\left.\begin{array}{r} 2xa_1 - b_1 = 0 \\ 2xa_2 + a_1^2 - b_2 = 0 \end{array}\right\} \begin{bmatrix} x & a_1 & a_2 \\ y & b_1 & b_2 \end{bmatrix} \mapsto \begin{bmatrix} x & a_1 \\ y & b_1 \end{bmatrix} \left\{ 2xa_1 - b_1 = 0. \right.$$

We shall see later exactly why this phenomenon is ensured by smoothness.

Example 2.25 (Cusp, continued). Let $\mathrm{X} := \{x^3 - y^2 = 0\}$, which has exactly one singular point at the origin $(0,0)$. What do the jet schemes look like at the preimages of certain points of X? Suppose throughout that $(x,y) \neq (0,0)$. Then

$$\pi_1^{-1}((x,y)) = \{(x, a_1, y, b_2) \in k^4 \mid 3x^2a_1 - 2yb_1 = 0\}$$

is the tangent line on X at the point (x,y). It resides in the (a_1, b_1)-plane.

On the other hand, at the singular point we have

$$\pi_1^{-1}((0,0)) = \{(0, a_1, 0, b_1) \in k^4\} \cong \mathbb{A}_k^2.$$

This distinct behaviour continues into higher-order jet schemes:

$$\begin{aligned}
\pi_2^{-1}((x,y)) &= \left\{ (x, a_1, a_2, y, b_1, b_2) \in k^6 \;\middle|\; \begin{array}{l} 3x^2a_1 - 2yb_1 = 0 \\ 3xa_1^2 + 3x^2a_2 - b_1^2 - 2yb_2 = 0 \end{array} \right\} \\
\pi_2^{-1}((0,0)) &= \left\{ (0, a_1, a_2, 0, b_1, b_2) \in k^6 \;\middle|\; b_1 = 0 \right\} \\
&= \left\{ (0, a_1, a_2, 0, 0, b_2) \in k^6 \right\} \\
&\cong \mathbb{A}_k^3.
\end{aligned}$$

The truncation morphism $\psi_{2,1}$ is surjective onto the fibers of nonsingular points of X:

$$\psi_{2,1} : \pi_2^{-1}((x,y)) \to \pi_1^{-1}((x,y))$$
$$\left.\begin{array}{r} 3x^2a_1 - 2yb_1 = 0 \\ 3xa_1^2 + 3x^2a_2 - b_1^2 - 2yb_2 = 0 \end{array}\right\} \begin{bmatrix} x & a_1 & a_2 \\ y & b_1 & b_2 \end{bmatrix} \mapsto \begin{bmatrix} x & a_1 \\ y & b_1 \end{bmatrix} \left\{ 3x^2a_1 - 2yb_1 = 0. \right.$$

Surjectivity here follows from the fact that for any $(x, a_1, y, b_1) \in \pi_1^{-1}((x,y))$, we can choose suitable $a_2, b_2 \in k$ such that $3xa_1^2 - b_1^2 = 3x^2a_2 - 2yb_2$. For any such choice of a_2, b_2, it follows that $(x, a_1, a_2, y, b_1, b_2) \in \pi_2^{-1}((x,y))$. Then clearly $\psi_{2,1}((x, a_1, a_2, y, b_1, b_2)) = (x, a_1, y, b_1)$.

However, for the fibers of the singular point it is a different story. Look at the following mapping:

$$\psi_{2,1} : \pi_2^{-1}((0,0)) \to \pi_1^{-1}((0,0))$$
$$\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & b_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix}$$

For this to be surjective, we require the image to be the whole (a_1, a_2)-plane. But instead, we get only the line $\{b_1 = 0\}$. Hence the truncation morphism $\psi_{2,1}$ is not surjective on the fibers of the singular point.

Smoothness—or lack thereof—has a big effect on both the jet schemes and the arc spaces. The examples given above serve merely as an illustration of this phenomenon, and a formal treatment will be given towards the end of the next section. Later in our discussion regarding the Nash problem, we shall see that this makes the theory of arcs a very robust tool in the study of singularities.

For now we have said everything that is necessary about jet schemes. For many of the important results concerning jet schemes, there are analogues for arc spaces too. It is better therefore to avoid this kind of repetition by advancing now to the setting of arc spaces.

Chapter 3

Arc spaces

We are now ready to properly learn about arcs, which means studying arc spaces. This chapter will collect a number of fundamental facts that will be necessary when we come to study the Nash problem.

> *Assumptions.* Unless otherwise stated, k is an algebraically closed field of arbitrary characteristic.

3.1 Existence

In this section we shall be using the notion of *direct* (resp. *inverse*) *limits* and *systems.* The details are not very complicated, so we omit them and instead offer the general references [ML98, §III] and [Lei14, §5].

The definition of the arc space is as simple as the following:

Definition 3.1. Let X be a scheme of finite type over k. The *arc space of* X is the k-scheme $X_\infty := \varprojlim X_m$.

Just like in the case of jet schemes, we shall be careful to ensure that this object exists. This time it is much more straightforward, but to avoid any possibility of doubt we shall be still be rigorous.

Lemma 3.2. *(a) Direct limits exist in the category of rings.*

(b) Suppose that $(X_m, \psi_{m,n})_{m \geq n}$ is an inverse system of affine schemes over k, such that $X_m = \operatorname{Spec} A_m$ for $A_m \in k\text{–}\mathcal{A}\mathrm{lg}$. Then $\varprojlim X_m = \operatorname{Spec}(\varinjlim A_m)$. In particular, the inverse limit exists in $\mathcal{S}\mathrm{ch}/k$ and is obviously affine.

Proof. See (a) [Mat86, Appendix A], (b) [EGA IV$_3$, Proposition 8.2.3]. □

Noting that affine schemes (of finite type over k) have affine jet schemes, we can make an inverse system out of them. The following is then immediate:

Proposition 3.3. *The arc space* X_∞ *exists, and is affine, for all affine schemes* X *of finite type over* k.

In general, when the arc space X_∞ exists, we also obtain canonical projection morphisms $\psi_m : X_\infty \to X_m$ for $m \in \mathbb{N}_0$. Denote the projection morphism ψ_0 into X by $\pi : X_\infty \to X$. Just like before, we will write $\psi_{X,m}$ and π_X if it would otherwise be ambiguous.

Definition 3.4. A morphism $f : X \to Y$ of schemes is said to be *affine* if there exists an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

Lemma 3.5. *Let* $(X_m, \psi_{m,n})_{m\geq n}$ *be an inverse system of schemes over* k *such that each* $\pi_{m,n}$ *is affine. Then the inverse limit* $X_\infty \in \mathcal{S}\mathrm{ch}/k$ *exists, and the corresponding projection morphisms* $\psi_m : X_\infty \to X_m$ *are also affine.*

Proof. See [Sta15, Tag 01YX]. □

The next task for this section is to justify existence of the arc space for schemes of finite type over k that are not necessarily affine. Combining the above lemma with the following result will suffice.

Proposition 3.6. *Let* X *be a scheme of finite type over* k. *Then the truncation morphisms* $\psi_{m,n}$ *form an inverse system* $(X_m, \psi_{m,n})_{m\geq n}$, *for* $m, n \in \mathbb{N}_0$. *Moreover, every truncation morphism is affine.*

Proof. For the first claim, just note that the $\psi_{m,n}$ are compatible in the sense that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & X_{m+1} \arrow[r, "\psi_{m+1,m}"] \arrow[rr, bend right, "\psi_{m+1,m-1}"'] & X_m \arrow[r, "\psi_{m,m-1}"] & X_{m-1} \arrow[r] & \cdots \end{tikzcd} \tag{3.1}$$

commutes for all $m \in \mathbb{N}$. This is readily verified by inspection of the polynomial truncation given in (2.13), and the definition of $\psi_{m,n}$ given in (2.14).

The next claim is also easy: Let $m, n \in \mathbb{N}_0$ such that $m \geq n$, and consider any open affine cover $\{U_i\}$ of X. We are going to show that $\psi_{m,n} : X_m \to X_n$ is an affine morphism. The open cover $\{\pi_n^{-1}(U_i) \overset{\text{L2.19}}{\cong} (U_i)_n\}$ of X_n is affine by Corollary 2.12. Then, again by Corollary 2.12,

$$\psi_{m,n}^{-1}((U_i)_n) = \psi_{m,n}^{-1}(\pi_n^{-1}(U_i)) \overset{(3.1)}{=} \pi_m^{-1}(U_i) \overset{\text{L2.19}}{\cong} (U_i)_m$$

is affine for each i. This proves the claim. □

Theorem 3.7. *The arc space* X_∞ *exists for all schemes* X *of finite type over* k.

Proof. Take the inverse system $(X_m, \psi_{m,n})_{m\geq n}$ of jet schemes of X and apply Lemma 3.5, which is allowed because of Proposition 3.6. □

Remark 3.8. While we have restricted our attention to arc spaces of schemes of finite type over an algebraically closed field, the notions of jet scheme and arc space hold in much higher generality. See [Vojo7] for what is probably the most thorough treatment of this matter.

Except for the final section, our existence result Theorem 3.7 will be sufficient for our discussion.

Unlike jet schemes, observe that the arc space is not necessarily of finite type over k. In fact, most of the time it certainly is not:

Proposition 3.9. *Let* X *be a scheme of finite type over* k*. If* $\dim X > 0$*, then* X_∞ *is not of finite type over* k*.*

Example 3.10 (Cusp, continued)**.** Let $X := \{x^3 - y^2 = 0\}$, and recall from Example 2.18 that

$$X_2 = \left\{ \begin{array}{c} (a_0 + a_1 t + a_2 t^2, \\ b_0 + b_1 t + b_2 t^2) \end{array} \in (k[t]/(t^3))^2 \;\middle|\; \begin{array}{l} a_0^3 - b_0^2 = 0 \\ 3a_0^2 a_1 - 2b_0 b_1 = 0 \\ 3a_0 a_1^2 + 3a_0^2 a_2 - b_1^2 - 2b_0 b_2 = 0 \end{array} \right\}.$$

It was also noted that X_3 will be defined by four equations, namely the three of X_2 and one other. Then the arc space X_∞ will be defined by infinitely many equations, obtained by the same procedure. Namely, we have

$$X_\infty = \left\{ \begin{array}{c} (a_0 + a_1 t + a_2 t^2 + \cdots, \\ b_0 + b_1 t + b_2 t^2 + \cdots) \end{array} \in (k[[t]])^2 \;\middle|\; \begin{array}{l} a_0^3 - b_0^2 = 0 \\ 3a_0^2 a_1 - 2b_0 b_1 = 0 \\ 3a_0 a_1^2 + 3a_0^2 a_2 - b_1^2 - 2b_0 b_2 = 0 \\ \vdots \end{array} \right\}.$$

3.2 The universal property

The arc space has a similar characterisation to the jet schemes.

Before proceeding, we need to know a little bit about formal schemes. This is somewhat of a technicality for us, but to not say anything could be confusing.

Let A be a ring and let I be an ideal of A. There is a natural inverse system of ring homomorphisms

$$\cdots \to A/I^3 \to A/I^2 \to A/I,$$

and the inverse limit $\hat{A} := \varprojlim_m A/I^m$ is called the *I-adic completion of* A.

Now to transport this to the language of schemes, take a Noetherian scheme X and let Y be a closed subscheme of X defined by a sheaf of ideals $\mathcal{I}$. The *formal completion of* X *along* Y is the locally ringed space $\mathfrak{X} := (Y, \mathcal{O}_{\mathfrak{X}})$, where $\mathcal{O}_{\mathfrak{X}} := \varprojlim_m \mathcal{O}_X/\mathcal{I}^m$. Then roughly speaking, a *formal scheme* is something that locally looks like a formal completion. Formal schemes are in general not schemes, and we denote the category of formal schemes over k by $\mathrm{f}\mathcal{S}\mathrm{ch}/k$.

In particular, if we take $Y = X$ in the process of formal completion, then $\mathcal{X} = X$. So the category of formal schemes contains all schemes, and accordingly the category of formal schemes over k contains all schemes over k. Moreover, one finds that formal schemes typically come up as direct limits of ordinary schemes, which explains the difference—the category of schemes is not closed under direct limits. Direct limits are how we shall meet formal schemes as well.

Note however that this description is only for Noetherian rings and schemes. Things get a bit technical, and there is not much to gain from discussing it in depth. By comparison, what we have said here is extremely brief—the point is to just get an idea for what is going on. A very nice introduction to this theory can be found in [Ill05].

So why do we care? Well, now we can do this:

Proposition 3.11. *Let* X *be a scheme of finite type over* k. *Then*

$$\mathrm{Hom}_{\mathrm{f}\mathcal{S}\mathrm{ch}/k}(Z, X_\infty) \cong \mathrm{Hom}_{\mathrm{f}\mathcal{S}\mathrm{ch}/k}(Z \hat{\times}_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]], X) \tag{3.2}$$

for all $Z \in \mathcal{S}\mathrm{ch}/k$. *Here,* $Z \hat{\times}_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$ *denotes formal completion of* $Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$ *along* $Z \times_{\mathrm{Spec}\, k} \{0\}$.

Proof. By Definition 2.10 we have an isomorphism of inverse systems

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, X_{m-1}) & \cong & \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^m), X) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, X_m) & \cong & \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X) \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, \varprojlim X_m) & \cong & \mathrm{Hom}_{\mathrm{f}\mathcal{S}\mathrm{ch}/k}(\varinjlim (Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1})), X) \\
\| & & \| \\
\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, X_\infty) & \cong & \mathrm{Hom}_{\mathrm{f}\mathcal{S}\mathrm{ch}/k}(Z \hat{\times}_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]], X),
\end{array}$$

which yields the desired result. □

We call (3.2) the *universal property of arc spaces.* This might seem a bit weird, but we can also reformulate this in the same way as was done in Remark 2.13: Namely, if $Z = \mathrm{Spec}\, A$ for some $A \in k\text{-}\mathcal{A}\mathrm{lg}$, then the universal property gives us

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\mathrm{Spec}\, A, X_\infty) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ch}/k}(\mathrm{Spec}\, A[[t]], X).$$

(See [Ish07, Remark 2.14] for further details.)

Definition 3.12. By virtue of the universal property, a morphism of (formal) schemes over k of the form

$$\alpha : Z\hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X$$

for $X, Z \in \mathcal{S}ch/k$ is called a *family of arcs on* X.

For such a family of arcs α, the scheme Z parametrises arcs on X in the following way: For each $z \in Z$, there exists a canonical morphism $i_z : \operatorname{Spec} k(z) \to Z$ with image z, where $k(z) \supseteq k$ is the residue field at z. Then composition with $\alpha : Z \to X_\infty$ gives a morphism

$$\alpha \circ i_z : \operatorname{Spec} k(z) \to X_\infty,$$

which by (3.2) corresponds to an arc

$$\alpha_z : \operatorname{Spec} k(z)[[t]] \to X.$$

Corollary 3.13. *For all schemes* X *of finite type over* k*, there exists a* universal family of arcs

$$X\hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X$$

induced by the identity map $\mathrm{Id}_{X_\infty} \in \mathrm{Hom}_{\mathcal{S}ch/k}$ *on the arc space of* X.

3.3 General properties

Having established the universal property of arc spaces, we now arrive at the fun part. In the remainder of this chapter we shall establish the fundamental theory regarding both jet schemes and arc spaces. Most of these facts arise very naturally, so the proofs are particularly pleasant to read.

Remark 3.14. Just like when we considered polynomial truncations in order to construct the truncation morphisms $\psi_{m,n}$, we can now look at the power series truncation

$$p^* : A[[t]] \to A[t]/(t^{m+1})$$
$$\sum_{i=0}^{\infty} \lambda_i t^i \mapsto \sum_{i=0}^{m} \lambda_i t^i$$

for any $A \in k\!-\!\mathcal{A}lg$ and obtain the scheme morphism

$$p : \operatorname{Spec} A[t]/(t^{m+1}) \to \operatorname{Spec} A[[t]].$$

The canonical projection morphisms are then precisely

$$\psi_m : X_\infty \to X_m$$
$$\alpha \mapsto \alpha \circ p$$

for all $\alpha : \operatorname{Spec} A[[t]] \to X$. In particular, if $A = K$ is a field extension of k, then $\pi(\alpha) = \alpha(0)$.

Proposition 3.15. *For all $m \in \mathbb{N}_0 \cup \{\infty\}$ there is a canonical isomorphism*

$$(X \times_{\operatorname{Spec} k} Y)_m \cong X_m \times_{\operatorname{Spec} k} Y_m$$

for all schemes X, Y of finite type over k

Proof. Let $Z \in \mathcal{S}\mathrm{ch}/k$ be arbitrary. Then

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, X_m \times_{\operatorname{Spec} k} Y_m) &\cong \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, X_m) \times \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, Y_m) \\
&\overset{(2.4)}{\cong} \begin{array}{c} \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X) \\ \times \\ \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), Y) \end{array} \\
&\cong \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X \times_{\operatorname{Spec} k} Y) \\
&\overset{(2.4)}{\cong} \operatorname{Hom}_{\mathcal{S}\mathrm{ch}/k}(Z, (X \times_{\operatorname{Spec} k} Y)_m)
\end{aligned}$$

for all $m \in \mathbb{N}_0$. If we replace $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})$ by $Z \hat{\times}_{\operatorname{Spec} k} k[[t]]$ in this argument, then we also prove the result for $m = \infty$ by application of the bijection (3.2). □

Proposition 3.16. *Let $f : X \to Y$ be a morphism of schemes of finite type over k. Then for every $m \in \mathbb{N}_0 \cup \{\infty\}$, a canonical morphism $f_m : X_m \to Y_m$ is induced such that the following diagram commutes:*

$$\begin{tikzcd}
X_m \arrow[r, "f_m"] \arrow[d, "\pi_{X,m}"'] & Y_m \arrow[d, "\pi_{Y,m}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{3.3}$$

Proof. Let $m \in \mathbb{N}_0$. It suffices to define the morphism f_m at the level of A-valued points for all $A \in k\text{-}\mathcal{A}\mathrm{lg}$:

$$\begin{aligned} f_m : X_m &\to Y_m \\ \alpha &\mapsto f \circ \alpha, \end{aligned}$$

regarding α as a morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X$. It is then obvious that the diagram (3.3) above commutes.

To obtain $f_\infty : X_\infty \to Y_\infty$, look at the following diagram:

$$\begin{tikzcd}
Y_\infty \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[r, "\psi_{Y,2,1}"] & Y_1 \arrow[r, "\psi_{Y,1,0}"] & Y \\
X_\infty \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r, "\psi_{X,2,1}"'] \arrow[u, "f_2"] & X_1 \arrow[r, "\psi_{X,1,0}"'] \arrow[u, "f_1"] & X \arrow[u, "f"]
\end{tikzcd} \tag{3.4}$$

From what we have already shown, it is clear that (3.4) commutes (cf. (2.14)). For each $m \in \mathbb{N}_0$, we obtain a morphism $f'_m := f_m \circ \pi_{X,m} : X_\infty \to Y_m$. Then the following diagram also commutes:

$$\begin{tikzcd} Y_\infty \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[r, "\varphi_{2,1}"] & Y_1 \arrow[r, "\varphi_{1,0}"] & Y \\ X_\infty \arrow[u, dotted, "\exists!\, f_\infty"] \arrow[urr, bend right, "f'_2"'] \arrow[urrr, bend right, "f'_1"'] \arrow[urrrr, bend right, "f'_0"'] \end{tikzcd} \tag{3.5}$$

Moreover, from the definition of inverse limit we obtain a unique morphism $f_\infty : X_\infty \to Y_\infty$ that is compatible in the sense of (3.5). Since $f'_0 = f \circ \pi_X$, it follows in particular that (3.3) commutes for $m = \infty$. This proves the result. □

Remark 3.17. Considering what we said in Remark 3.14, it is clear that a suitable candidate for f_∞ is

$$\begin{aligned} f_\infty : X_\infty &\to Y_\infty \\ \alpha &\mapsto f \circ \alpha. \end{aligned}$$

Since f_∞ is unique, this is the *only* candidate and hence the formal definition.

Corollary 3.18. *(a) For all $m \in \mathbb{N}_0$,*

$$\begin{aligned} \mathcal{S}\mathrm{ch}_{\mathrm{ft}}/k &\to \mathcal{S}\mathrm{ch}_{\mathrm{ft}}/k \\ X &\mapsto X_m \end{aligned}$$

is a functor.

(b) Moreover,

$$\begin{aligned} \mathcal{S}\mathrm{ch}_{\mathrm{ft}}/k &\to \mathcal{S}\mathrm{ch}/k \\ X &\mapsto X_\infty \end{aligned}$$

is a functor.

Definition 3.19. A morphism of schemes $f : X \to Y$ is said to be *formally étale* if the following property holds: Let Z be a scheme and suppose that $Z_0 \subseteq Z$ is a closed subscheme defined by a locally nilpotent sheaf of ideals $\mathcal{J}$ of Z. Then for any diagram of the form

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d] & Z \arrow[d] \arrow[dl, dashed, "\exists!"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.6}$$

there exists a unique morphism $Z \to X$ such that the whole diagram commutes.

Remark 3.20. The above definition is sometimes given with the stronger hypothesis that Z is affine and that $\mathcal{J}$ is nilpotent. The two definitions are equivalent; see [EGA IV$_4$, Définition 17.1.1 and Remarques 17.1.2(iv)].

Definition 3.21. A morphism of schemes is said to be *étale* if it is formally étale and locally of finite presentation.

Remark 3.22. A morphism $f : X \to Y$ of schemes of finite type over k is étale if, and only if, it is formally étale. This follows from the fact that k is (trivially) Noetherian, in which case f is of finite type if, and only if, it is of finite presentation (cf. [EGA IV$_1$, §1.4]). In particular it is locally of finite presentation.

Proposition 3.23. *Suppose that $f : X \to Y$ is an étale morphism of schemes of finite type over k. Then $X_m \cong Y_m \times_Y X$ for every $m \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. First let $m \in \mathbb{N}_0$. Consider the situation described in the commutative diagram

$$\begin{tikzcd} Z \arrow[dr, dotted, "\exists!?"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & X_m \arrow[r, "f_m"] \arrow[d, "\pi_{X,m}"'] & Y_m \arrow[d, "\pi_{Y,m}"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.7}$$

for any suitable $Z \in \mathcal{S}ch/k$. The lower square commutes by Proposition 3.16. It suffices to prove that there exists a unique morphism $Z \to X_m$ that maintains the commutativity of the diagram. To give such a morphism is by Definition 2.10 the same as giving a morphism

$$Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to X.$$

In exactly the same way, we also regard the existing morphism $q_2 : Z \to Y_m$ as $q_2 : Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to Y$.

We have reformulated the situation as described by the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "q_1"'] & Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \arrow[d, "q_2"] \arrow[dl, dashed, "\exists!"] \\ X \arrow[r, "\text{ét}", "f"'] & Y \end{tikzcd} \tag{3.8}$$

This of course requires some explanation:

- By the definition of f_m (cf. Proposition 3.16) and the above rewriting of q_2, we use the bottom triangle of (3.8) to verify that the top triangle of (3.7) commutes.

- Similarly, the top triangle of (3.8) corresponds to the left triangle of (3.7). To see this, examine the affine case. For $Z = \operatorname{Spec} A$, $A \in k\text{–}\mathcal{A}\mathrm{lg}$, we have $\pi_{X,m}(\alpha) = \alpha \circ p$ for $p : \operatorname{Spec} A \to \operatorname{Spec} A[t]/(t^{m+1})$. Then note that $i = p$.

- Consider an open affine subscheme U of Z. Since $U \cong U \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t)$, it is defined by a sheaf of ideals

$$\mathcal{J} := \mathcal{O}_{U \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})}/(t)$$

of $U \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1})$. Then $\mathcal{J}^{m+1} = 0$, so $\mathcal{J}$ is a nilpotent ideal (of order m). It follows that Z is defined by a locally nilpotent ideal of $Z \times_{\operatorname{Spec} k} k[t]/(t^{m+1})$.

Since f is étale, it is formally étale. Hence the diagram (3.8) is of the form (3.6), and there exists a unique morphism $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to X$ such that the whole diagram (3.8) commutes. Then (3.7) commutes, and this proves the claim for $m \in \mathbb{N}_0$.

In the case $m = \infty$, just note that

$$X_\infty = \varprojlim X_m \cong \varprojlim (Y_m \times_Y X) = (\varprojlim Y_m) \times_Y X \cong Y_\infty \times_Y X. \qquad \square$$

The following result is an enhancement of our previous Lemma 2.19.

Corollary 3.24. *If* $U \hookrightarrow X$ *is an open immersion of schemes of finite type over* k*, then* $U_m \cong \pi_{X,m}^{-1}(U)$ *for all* $m \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. An open immersion of schemes is étale (see [EGA IV$_4$, Théorème 17.9.1]). Hence $U_m \overset{\text{P3.23}}{\cong} X_m \times_X U \cong \pi_{X,m}^{-1}(U)$. $\square$

Corollary 3.25. *If* $\iota : Z \hookrightarrow X$ *is an open (resp. closed) immersion of schemes of finite type over* k*, then the induced morphism* $\iota_\infty : Z_\infty \hookrightarrow X_\infty$ *is an open (resp. closed) immersion for all* $m \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. The open case follows from Corollary 3.24. For the closed case, suppose that X is affine and that it is defined by some $(f_1, \dots, f_r)$ for $r \in \mathbb{N}$. Then Z will be defined by $(f_1, \dots, f_r, \dots, f_u)$ for $u \in \mathbb{N}$, $u \geq r$. Now recall (cf. Proposition 2.11) that X_m is defined by suitable $(\{F_{\ell s}\}_{\ell=1,s=0}^{\ell=r,s=m})$, and similarly Z_m will be defined by $(\{F_{\ell s}\}_{\ell=1,s=0}^{\ell=u,s=m})$. Hence Z_m is a closed subscheme of X_m. Observe that this works not only for all $m \in \mathbb{N}_0$, but also $m = \infty$ (following the same logic as Example 3.10). $\square$

Proposition 3.26. *Let $f : X \to Y$ be a proper morphism of schemes of finite type over k. If the restriction of f induces an isomorphism*

$$X \setminus V \cong Y \setminus W$$

for closed subsets $V \subseteq X$ and $W \subseteq Y$, then the restriction of f_∞ induces a bijection

$$X_\infty \setminus V_\infty \leftrightarrows Y_\infty \setminus W_\infty. \tag{3.9}$$

Proof. Pick up an arc $\alpha \in Y_\infty \setminus W_\infty$. If $\alpha(0) \in Y \setminus W$ then

$$\alpha \in \pi_Y^{-1}(Y \setminus W) \overset{C3.24}{\cong} (Y \setminus W)_\infty \overset{f_\infty}{\cong} (X \setminus V)_\infty,$$

showing bijectivity of f_∞ between subsets $(Y \setminus W)_\infty$ and $(X \setminus V)_\infty$.

On the other hand, if $\alpha(0) \in W$ then $\alpha(\eta) \in Y \setminus W \cong X \setminus V$. So the restriction of α to the generic point corresponds to a morphism $\operatorname{Spec} K((t)) \to X$, and we can form a commutative diagram as follows:

$$\begin{tikzcd} \operatorname{Spec} K((t)) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} K[[t]] \arrow[r, "\alpha"'] \arrow[ur, dotted, "\tilde{\alpha}"] & Y \end{tikzcd}$$

As $K[[t]]$ is a valuation ring (cf. [Mat86, §10]) over the field $K((t))$, we may apply the *valuative criterion of properness* (see [Har77, Theorem II.4.3]; we will not state it here). It tells us that for the commutative diagram above, there exists a unique morphism $\tilde{\alpha} : \operatorname{Spec} K[[t]] \to X$ that makes the whole diagram commute. In particular, $\alpha = f \circ \tilde{\alpha} \overset{R3.17}{=} f_\infty(\tilde{\alpha})$. It is also evident from the commutative diagram that $\tilde{\alpha} \in (X_\infty \setminus V_\infty) \setminus (X \setminus V)_\infty$, whence the bijection. □

Remark 3.27. The map (3.9) is continuous, since it is a restriction of f_∞ to an open subset (cf. Corollary 3.25). Indeed it is also a morphism of schemes for the same reason, but this fact is not of any importance to us. Note however that the inverse map is *not* necessarily continuous.

3.4 Affine space, smoothness, and irreducibility

This section collects some important results regarding smoothness and irreducibility. They will be particularly important in the following chapters, when we come to study the Nash problem.

Example 3.28 (Affine space, continued). Let $X := \mathbb{A}_k^n$ for $n \in \mathbb{N}_0$. Then $X_\infty = \operatorname{Spec} k[\{x_{ij}\}_{i=1,j=0}^{i=n,j=\infty}] \cong \mathbb{A}_k^\infty$.

Warning 3.29. Although the set of closed points of $\mathbb{A}_k^n$ corresponds to k^n for $n \in \mathbb{N}_0$, this property does not hold in general for $n = \infty$. That is to say, the set of closed points of $\mathbb{A}_k^\infty$ does not necessarily correspond to the set $k^\infty := \{(a_1, a_2, \dots) \mid a_i \in k \ \ (i \in \mathbb{N})\}$.

Proposition 3.30. *Every closed point of $\mathbb{A}_k^\infty$ is a k-valued point if, and only if, the field k is uncountable.*

Proof. See [Ish04, Proposition 2.10 and Proposition 2.11]. □

Since the only condition we are placing on the field k is that it is algebraically closed, it is conceivable that k will be countable. But in spite of this drawback, we can still do perfectly good mathematics with the arc spaces.

Definition 3.31. A morphism of schemes $f : \mathrm{X} \to \mathrm{Y}$ is said to be *(Zariski) locally trivial with fiber* F (for $\mathrm{F} \in \mathcal{S}\mathrm{ch}$) if for every point in Y there exists an open neighbourhood $\mathrm{U} \subseteq \mathrm{Y}$ such that $f^{-1}(\mathrm{U}) \cong \mathrm{U} \times \mathrm{F}$, where the restriction of f to $f^{-1}(\mathrm{U})$ corresponds to the projection $\mathrm{U} \times \mathrm{F} \to \mathrm{U}$. In this case, we also say that f is a *locally trivial fibration.*

Lemma 3.32 (Pasting of pullbacks). *Consider a commutative diagram*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(a)}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(b)}"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

where (a) *(resp.* (b)*) denotes the left (resp. right) inner square and* (ab) *denotes the outside rectangle. Suppose that* (b) *is a pullback. Then* (a) *is a pullback if, and only if,* (ab) *is a pullback.*

Proof. If (b) and (ab) are pullbacks, then it is obvious that (a) is a pullback. Conversely, suppose that (a) and (b) are pullbacks. Then the proof is indicated by the following diagram:

$$\begin{tikzcd} \cdot \arrow[dr, dotted, "\exists!"] \arrow[drr, dotted, "\exists!"] \arrow[drrr, bend left] \arrow[ddr, bend right] \arrow[dr, phantom, "\text{(a)}", swap] & & & \\ & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(a)}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(b)}"] & \cdot \arrow[d] \\ & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

□

Proposition 3.33. *Suppose that* X *is a smooth scheme of finite type over k, and that* X *has dimension $n \in \mathbb{N}_0$. Then the truncation morphisms $\psi_{m+1,m} : \mathrm{X}_{m+1} \to \mathrm{X}_m$ are locally trivial with fiber $\mathbb{A}_k^n$ for all $m \in \mathbb{N}_0$.*

Proof. First note that X is smooth at a point $x \in X$ if, and only if, there exists an open neighbourhood $U \subseteq X$ of x and an étale morphism $U \to \mathbb{A}^r_k$ for some $r \in \mathbb{N}_0$ (by [EGA IV$_4$, Corollaire 17.11.4]). Additionally, since $\dim X = n$ we can require that $r = n$ (cf. [EGA IV$_4$, Proposition 17.15.5]).

With that in mind, pick up a point $x \in X$. Since X is smooth here, there exists an open affine neighbourhood $U \subseteq X$ of x and an étale morphism $U \to \mathbb{A}^n_k$. Now look at the following commutative diagram:

$$
\begin{tikzcd}
U_{m+1} \arrow[r] \arrow[d, "\psi_{m+1,m}"'] \arrow[dr, phantom, "(b)"] & (\mathbb{A}^n_k)_{m+1} \arrow[r, phantom, "\overset{E2.23}{\cong}"] \arrow[d] & \mathbb{A}^{n(m+1)}_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k \arrow[r] \arrow[dr, phantom, "(c)"] & \mathbb{A}^n_k \arrow[d] \\
U_m \arrow[r] \arrow[d, "\pi_m"'] \arrow[dr, phantom, "(a)"] & (\mathbb{A}^n_k)_m \arrow[r, phantom, "\underset{E2.23}{\cong}"] \arrow[d] & \mathbb{A}^{n(m+1)}_k \arrow[r] & \operatorname{Spec} k \\
U \arrow[r, "\text{ét}"] & \mathbb{A}^n_k & &
\end{tikzcd}
$$

At this point the proof begins to write itself.

Claim. $U_{m+1} \cong U_m \times_{\operatorname{Spec} k} \mathbb{A}^n_k$.

Proof of claim. Clearly (c) is a fiber product by construction. Then we have a double pasting[1] of pullbacks, so

$$
\begin{tikzcd}
& {[a]} \arrow[dr, Rightarrow] & & \\
P3.23 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \arrow[r, phantom, "L3.32"] & {[b]} \arrow[dr, Rightarrow] & \\
& {[ab]} \arrow[ur, Rightarrow] & \arrow[r, phantom, "L3.32"] & {[bc]} \\
& & {[c]} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

where [a] (and so on) means "(a) is a fiber product". ■

Then

$$\psi^{-1}_{m+1,m}(U_m) \overset{C3.24}{\cong} U_{m+1} \cong U_m \times_{\operatorname{Spec} k} \mathbb{A}^n_k \overset{C3.24}{\cong} \pi_m^{-1}(U) \times_{\operatorname{Spec} k} \mathbb{A}^n_k,$$

and the restriction of $\psi_{m+1,m}$ to this preimage yields the desired projection onto $\pi_m^{-1}(U)$. Since we can construct an open cover of X_m using subschemes of the form $\pi_m^{-1}(U)$, it follows that $\psi_{m+1,m}$ is locally trivial with fiber $\mathbb{A}^n_k$. □

[1] A collage?

Remark 3.34. Keep the hypothesis of Proposition 3.33. For every point in X we obtain an étale morphism $U \to \mathbb{A}_k^n$. It is clear from the discussion in the above proof that

$$\pi_m^{-1}(U) \cong U \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn}$$

for all $m \in \mathbb{N}_0$. This extends to $\pi^{-1}(U)$, as we shall now demonstrate. We have the following inverse systems:

$$\begin{tikzcd}
\mathbb{A}_k^{\infty} \arrow[r] & \cdots \arrow[r] & \mathbb{A}_k^{mn} \arrow[r] & \cdots \arrow[r] & \operatorname{Spec} k \\
\varprojlim (U \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn}) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \cdots \arrow[r] & U \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn} \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \cdots \arrow[r] & U \arrow[u] \\
\pi^{-1}(U) & & \pi_m^{-1}(U) & &
\end{tikzcd}$$

In this situation, [EGA IV$_3$, Proposition 8.2.5] states that $\varprojlim_m (U \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn}) = U \times_{\operatorname{Spec} k} \varprojlim_m \mathbb{A}_k^{mn} = U \times_{\operatorname{Spec} k} \mathbb{A}_k^{\infty}$. In particular, we have that

$$\pi^{-1}(U) \cong U \times_{\operatorname{Spec} k} \mathbb{A}_k^{\infty}.$$

Corollary 3.35. *If X is a smooth scheme of finite type over k, then the truncation morphisms $\psi_{m,n} : X_m \to X_n$ and the projection morphisms $\psi_m : X_\infty \to X_m$ are surjective for all $m, n \in \mathbb{N}_0$, $m \geq n$.*

Proof. It is quite clear from the definition that a locally trivial fibration is surjective. We just showed that under the given hypothesis, all truncation morphisms are locally trivial fibrations; therefore they are all surjective.

For the latter claim, let $m \in \mathbb{N}_0$. The projection morphism ψ_m is surjective if, and only if, the truncation morphisms $\psi_{m',m}$ are surjective for all $m' \in \mathbb{N}_0$, $m' \geq m$. (This is a general fact about inverse systems of schemes; see [EGA IV$_3$, Proposition 8.3.8(i)].) Hence all the projection morphisms are surjective. □

In the context of Corollary 3.35, note in particular that this means all of the projection morphisms $\pi_m : X_m \to X$ and $\pi : X_\infty \to X$ are surjective.

Proposition 3.36. *Suppose that X is a smooth scheme of finite type over k. Then for any irreducible subscheme $Z \subseteq X$, $\pi^{-1}(Z)$ and $\pi_m^{-1}(Z)$ are irreducible for all $m \in \mathbb{N}_0$.*

Proof. Let $m \in \mathbb{N}_0$. Suppose for contradiction that $\pi_m^{-1}(Z) = E_1 \cup E_2$, for distinct nontrivial closed subsets $E_1, E_2 \subseteq \pi_m^{-1}(Z)$.

For any point of $Z \subseteq X$ there exists an open neighbourhood $U \subseteq X$ and an étale morphism $U \to \mathbb{A}_k^r$ for some $r \in \mathbb{N}_0$. Set $V := U \cap Z$, which will also be irreducible and open in Z. The preimage $\pi_m^{-1}(V)$ must intersect either E_1, E_2, or both.

If $\pi_m^{-1}(V)$ intersects both then it is not irreducible. This can't happen though: The fiber product diagram

$$\begin{tikzcd} \pi_m^{-1}(V) \arrow[r, hook] \arrow[d, "\pi_m"'] & \pi_m^{-1}(U) \arrow[d, "\pi_m"] & \overset{R3.34}{\cong} & U \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn} \\ V \arrow[r, hook] & U \end{tikzcd}$$

shows that $\pi_m^{-1}(V) \cong V \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn}$. This is a fiber product of irreducible schemes and (by [EGA IV$_2$, Corollaire 4.5.8(i)]) is therefore itself irreducible. Hence $\pi_m^{-1}(V)$ intersects only E_1 or E_2.

Moreover, since Z is irreducible it has a unique generic point η which is also an element of every open subset of Z. So $\eta \in V$ and therefore (without loss of generality) we can say that $\pi_m^{-1}(V) \subseteq E_1$ and $\eta \notin \pi_m(E_2)$.

Since such a neighbourhood U exists for all points in X, it follows that $E_1 \cap E_2 = \emptyset$ and we conclude that $E_2 = \emptyset$. Contradiction ※. Therefore $\pi_m^{-1}(Z)$ is irreducible.

The argument for $\pi^{-1}(Z)$ is exactly the same—we just set $m = \infty$. When we say that $\pi^{-1}(U) \cong U \times_{\operatorname{Spec} k} \mathbb{A}_k^\infty$ is a product of irreducible schemes, it is worth emphasising that $\mathbb{A}_k^\infty$ is indeed irreducible. This follows from the fact that the ring $k[x_1, x_2, \dots]$ is integral. Then $\mathbb{A}_k^\infty = \operatorname{Spec} k[x_1, x_2, \dots]$ is an integral scheme (cf. [EGA I, Proposition 5.1.4]) and hence irreducible. □

Example 3.37. Consider $X = \{xy + z^2 = 0\} \subseteq \mathbb{A}_k^3$, which has only one singular point at 0. A simple computation shows that

$$X_2 = \left\{ \begin{array}{l} a_0 b_0 + c_0^2 = 0 \\ a_1 b_0 + a_0 b_1 + 2c_0 c_1 = 0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 + c_1^2 + 2c_0 c_2 = 0 \end{array} \right\} \subseteq \mathbb{A}_k^9.$$

We claim that X_2 is irreducible. To see this, first note that $X \setminus \{0\}$ is a smooth scheme and so by Proposition 3.36, $\pi_2^{-1}(X \setminus \{0\})$ will be an irreducible subscheme of X_2 of dimension 6 (cf. Proposition 3.33). Moreover, one can compute that $\pi_2^{-1}(\{0\})$ defines a hypersurface in $\mathbb{A}_k^6$ (compare with Example 2.25), i.e. it will have dimension 5. But the irreducible components of X_2 must have dimension ≥ 6, since X_2 is defined by three equations in $\mathbb{A}_k^9$. Therefore $\pi_2^{-1}(\{0\})$ does not contribute any irreducible component of X_2, meaning that X_2 must have precisely one irreducible component—namely, the closure of $\pi_2^{-1}(X \setminus \{0\})$.

This concludes our preliminary discussion regarding the theory of arcs. This will put is in good stead for the following chapters, which in a sense can be seen as a glorified application of this theory. Indeed we shall repeatedly appeal to the results proven so far, and in the end pratically everything we have said will have been used in one way or another.

Part II

The Nash problem

Chapter 4

The Problem

Assumptions. Unless otherwise stated, k is an algebraically closed field of arbitrary characteristic.

4.1 Preliminaries

The Nash problem is one of varieties—I do not know of a fully scheme-theoretic analogue. However, as we pointed out in Proposition 3.9, the arc space is almost never a variety. Our discussion will therefore continue to be played out in the realm of schemes, where by "variety" we just mean a scheme with the following nice properties:

Definition 4.1. A *variety* is an integral separated scheme of finite type over an algebraically closed field k.

But the varieties we are concerned with are not exactly perfect, in the sense that they are singular.

Definition 4.2. Let X be a scheme of finite type over k and let $x \in \mathrm{X}$ be a point. Then X is said to be *regular* (or *nonsingular*) *at* x if the local ring $\mathcal{O}_{\mathrm{X},x}$ is regular (cf. [Mat86, §19]). If $\mathcal{O}_{\mathrm{X},x}$ is not regular, then X is said to be *singular at* x.

If X is regular (resp. singular) at all points, then X is said to be *regular* (resp. *singular*).

For schemes of finite type over k, regularity coincides with the otherwise stronger property of *smoothness*. Formally:

Proposition 4.3. *Let* X *be a scheme of finite type over* k. *A point* $x \in \mathrm{X}$ *is smooth if, and only if, it is regular.*

Proof. Finite type over k is equivalent to finite presentation over k (cf. Remark 3.22). The claim is then equivalent to [EGA IV$_4$, Théorème 17.5.1].[1] □

[1] In fact, the hypothesis can be weakened to *locally* of finite type.

The definition of smoothness in full generality is somewhat involved, so we shall use the above equivalence as our definition of smooth—we won't be using it more generally anyway.

Definition 4.4. Let X be a scheme of finite type over k. The *singular locus* of X is the set $\operatorname{Sing} X$ of singular points in x.

Proposition 4.5. *The singular locus of a scheme X of finite type over k is a proper closed subset of X.*

Proof. In practice we will only care about this property for varieties; see [Har77, Theorem II.8.16] for that. The general result is due to Zariski; see [EGA IV_4, Théorème 17.5.1]. □

By virtue of closedness we can endow $\operatorname{Sing} X$ with a scheme structure. This allows us to consider its arc space $(\operatorname{Sing} X)_\infty$, which is essential in the construction of the Nash map. Note that we have a closed immersion $(\operatorname{Sing} X)_\infty \hookrightarrow X_\infty$ (cf. Corollary 3.25).

Now, for two basic topological notions.

Definition 4.6. Let X be a nonempty topological space.

(a) The space X is said to be *irreducible* if it possesses the following property: when $X = E_1 \cup E_2$ for closed subsets $E_1, E_2 \subseteq X$, it follows that $E_1 = X$ or $E_2 = X$.

(b) A subset $Z \subseteq X$ is said to be an *irreducible component of* X if it is a maximal irreducible subset of X, i.e. Z is not strictly contained in any proper irreducible subset of X.

We also use this terminology for schemes when their underlying topological space exhibits the given property.

Next, some fundamental facts regarding this definition now follow:

Lemma 4.7. *In the following, "topological space" means "Zariski topological space".*

(a) A topological space is the union of its irreducible components.

(b) A scheme of finite type over k has finitely many irreducible components.

(c) An irreducible component is always closed.

(d) An irreducible topological space has a unique generic point.

(e) Any open subset of an irreducible topological space X must contain the generic point of X.

Proof. See [EGA 0_I, §2]. □

In our discussion we shall use the above results freely and without explicit mention.

4.2 Transversal arcs and Nash components

It turns out that the arc space reveals a great deal about the nature of a variety's singularities. We would like to examine the parts of the arc space of X that are in a sense transversal to the singular locus. Consider the following definition:

Definition 4.8. Let X be a singular variety. An arc on X whose image has nonempty intersection with both $\operatorname{Sing} X$ and $X \setminus \operatorname{Sing} X$ is said to be *transversal* (with respect to the singular locus of X).

Since the complement of $\operatorname{Sing} X$ is open in X, it follows that its preimage $\pi^{-1}(X \setminus \operatorname{Sing} X)$ under the projection morphism will be isomorphic to the arc space $(X \setminus \operatorname{Sing} X)_\infty$ (cf. Corollary 3.24). In particular, there exists no arc on X that is centered outside the singular locus but whose generic point is mapped into a singular point. Put another way, we can only "measure" the singularities of X by considering arcs through its singular points (i.e. contained in $\pi^{-1}(\operatorname{Sing} X)$).

The insight that Nash had (cf. [Nas95]) was to consider irreducible components of $\pi^{-1}(\operatorname{Sing} X)$ with this kind of transversal property.

Definition 4.9. Let X be variety and consider the preimage

$$\pi^{-1}(\operatorname{Sing} X) = \{ \text{ arcs on X through } \operatorname{Sing} X \ \}.$$

An irreducible component of $\pi^{-1}(\operatorname{Sing} X)$ is called a *Nash component* (with respect to X) if it contains a transversal arc α.

Remark 4.10. Nash components are also known as *good components*, for example in [IK03].

Figure 4.1 gives a picture of what is going on here. We have considered two irreducible components C and C' of $\pi^{-1}(\operatorname{Sing} X)$ and are looking at their images on X. The closed points (denoted by a dots $\bullet$) of each arc are necessarily mapped to the singular locus. The generic points (denoted by squiggles) could land anywhere.

The component C has two arcs whose generic points are mapped into $X \setminus \operatorname{Sing} X$. If we pick up one such arc α and take the closure $\overline{\alpha(\eta)}$ in X, we will recover the image $\alpha(0)$ of the closed point too. So the closed point acts as a kind of anchor with the generic point encircling it. These two arcs are transversal, so in particular C is a Nash component.

On the other hand, the component C has no transversal arcs; it is totally contained in $(\operatorname{Sing} X)_\infty$. This is an example of an irreducible component of $\pi^{-1}(\operatorname{Sing} X)$ that is not a Nash component.

In fact, since X is irreducible it follows that transversal arcs map the generic point of $\operatorname{Spec} K[[t]]$ to the generic point of X.

Proposition 4.11. *Let* X *be a variety. If an arc* $\alpha : \operatorname{Spec} K[[t]] \to X$ *is transversal, then the induced ring homomorphism* $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ *is injective.*

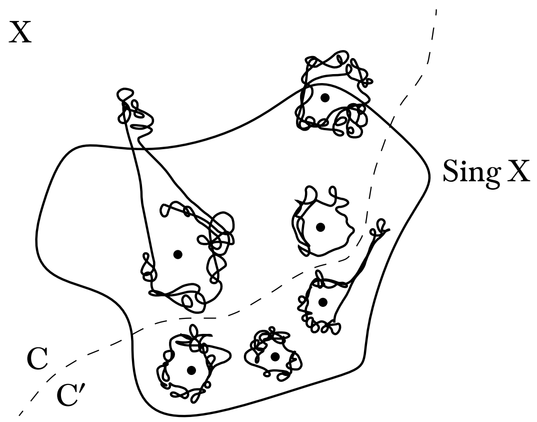


Figure 4.1: Visualisation of two irreducible components of $\pi^{-1}(\operatorname{Sing} X)$ mapped onto X. Here, C is a Nash component but C′ is not.

Proof. Suppose without loss of generality that $X = \operatorname{Spec} A$ for some suitable k-algebra A and suppose that α is transversal. By continuity of the morphism α, we see that the generic point η of $\operatorname{Spec} K[[t]]$ must map to the generic point of X. Hence the induced ring homomorphism $\alpha^* : A \to K[[t]]$ will be injective, and in particular so too the ring homomorphism $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ will be injective. □

The notion of a Nash component is not a redundant distinction, as evidenced by the following example:

Example 4.12 ([IK03, Example 2.13]). Suppose that k has characteristic $p > 0$ and consider the surface $X := \{x^p - y^p z\} \subseteq \mathbb{A}^3_k$ with singular locus $\operatorname{Sing} X = \{x = y = 0\}$. There exists an irreducible $C' \subseteq (\operatorname{Sing} X)_\infty$ that is not contained in the closure $\overline{X_\infty \setminus (\operatorname{Sing} X)_\infty}$, from which it follows that C′ is an irreducible component of $\pi^{-1}(\operatorname{Sing} X)$. Therefore C′ is not a Nash component.

On the other hand, in characterstic zero the situation is much simpler:

Proposition 4.13. *Suppose that k has characteristic zero and consider a variety* X *over k. Every irreducible component of* $\pi^{-1}(\operatorname{Sing} X)$ *is a Nash component.*

Proof. This is an immediate consequence of [IK03, Lemma 2.12], which states that every arc through Sing X is a specialisation of a transversal arc. □

This gives one side of the Nash map. Before discussing the other side and the map's construction, we need a bit more background theory.

4.3 Some birational geometry

There are two closely related ideas to take away from this section: that of birationality between varieties, and of resolution of singularities. We are only giving a brief overview here—for further reading, see [Kol07].

Definition 4.14. Let X and Y be schemes.

(a) Consider two open dense subsets $U, V \subseteq X$. Two morphisms $f : U \to Y$ and $g : V \to Y$ are said to be *equivalent* if $f|_W = g|_W$ for some $W \subseteq U \cap V$ dense and open in X. *Note that this is an equivalence relation.*

(b) A *rational map* from X to Y is an equivalence class of the equivalence relation (a) and is denoted by a dashed arrow $X \dashrightarrow Y$.

(c) A morphism that is contained in the equivalence class of a rational map f is said to be a *representative morphism of f*.

(d) A rational map $f : X \dashrightarrow Y$ is said to be *defined at a point $x \in X$* if there exists a representative morphism $U \to Y$ of f such that $x \in U$. The set of all such points x is called the *domain of definition of f*.

Remark 4.15. (a) The use of terminology and notation for Definition 4.14(b) is a little inconsistent between authors. Some simply say *map* to mean a rational map (e.g. [KM98], [Mat02]). Others denote rational maps using an ordinary full arrow (e.g. [Har77]), or forgo the use of any arrow at all (e.g. [EGA I]). Our choice is consistent with [Kol07].

(b) A rational map $f : X \dashrightarrow Y$ does not necessarily admit a representative morphism $U \to Y$ where U is equal to the domain of definition of f.

In general we cannot compose arbitrary rational maps: Pick up two rational maps f and g. It is conceivable that the image of all representative morphisms of f have empty intersection with the domain of definition of g. In that case a composition $g \circ f$ would not be possible. But, with some additional conditions we can ensure the compatibility of rational maps.

Definition 4.16. A rational map of irreducible schemes is said to be *dominant* if it admits a dominant representative morphism.

Actually, we get something a little stronger from this definition.

Proposition 4.17. *Suppose* $X \dashrightarrow Y$ *is a dominant rational map of irreducible schemes. Then all its representative morphisms are dominant.*

Proof. Denote the (unique) generic point of X (resp. Y) by η (resp. ξ). Suppose that f is a suitable dominant representative morphism. Let $V \subseteq Y$ be an open subset of Y, which necessarily will contain ξ. Then $\xi \in \mathrm{Im}(f)$ and

$\eta \in f^{-1}(V)$. Hence $f|_{f^{-1}(V)}$ is also a dense representative morphism. Any other representative morphism must agree with $f|_{f^{-1}(V)}$ on some dense open subset of $f^{-1}(V)$, so in particular on the generic point η. This makes the image of all representative morphisms contain ξ, and therefore they are all dominant. □

Composition of such rational maps is then guaranteed by virtue of the fact that the intersection of the image of a rational map $X \dashrightarrow Y$ will have nonempty intersection with the domain of definition of another rational map $Y \dashrightarrow Z$.

Having demonstrated some of the machinery of rational maps, we don't need to be so general anymore. Since we shall only be considering rational maps between varieties, the rest of this discussion on birational geometry will be in the context of varieties.

Definition 4.18. (a) A rational map $f : X \dashrightarrow Y$ between varieties is said to be *birational* if it has a rational inverse, i.e. a rational map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$ as rational maps.

(b) Two varieties X and Y are said to be *birational* (or *birationally equivalent*, if speaking verbosely) if there exists a birational map $f : X \dashrightarrow Y$.

Remark 4.19. We can make a category with varieties as objects and rational maps as morphisms. Then isomorphisms in this category are the birational maps of Definition 4.18(a), and to say that two objects are isomorphic is equivalent to saying they are birational.

Definition 4.20. A morphism $X \to Y$ of varieties is said to be *birational* if it is a birational map.

There's no trickery in this definition—it serves merely to emphasise that rational maps can of course also be morphisms.

Remark 4.21. Let $f : X \to Y$ be a birational morphism of varieties. There must necessarily exist some maximal nonempty open subset $V \subseteq Y$ such that f restricted to $f^{-1}(V)$ induces an isomorphism.

Indeed, there is a very special kind of birational morphism:

Definition 4.22. Consider a singular variety X. A *resolution of singularities of* X is a proper birational morphism $f : Y \to X$ from a nonsingular variety Y such that f restricts to an isomorphism $Y \setminus f^{-1}(\operatorname{Sing} X) \cong X \setminus \operatorname{Sing} X$.

For clarity, a resolution of singularities may simply be called a *resolution*. However to avoid any confusion, we won't use this shorthand in the statement of any definition, proposition, and so on.

A resolution being proper is of extreme value. Indeed, we shall use this directly in conjunction with Proposition 3.26 to build the Nash map.

4.4 Construction of the Nash map

Now we are going to construct the Nash map.

> *Assumptions.* X is a singular variety, and resolutions of singularities of X exist.

Let $f : Y \to X$ be a resolution of singularities of X. By Definition 4.22 we have an isomorphism

$$Y \setminus f^{-1}(\operatorname{Sing} X) \cong X \setminus \operatorname{Sing} X$$

between the nonsingular part of X and its preimage under f. The construction involves looking at arcs centered outside this part of X and Y, namely the singular locus of X and its preimage under f.

Let $\{E_j\}_{j=1}^{m}$ be the irreducible components of $f^{-1}(\operatorname{Sing} X)$ (for suitable $r \in \mathbb{N}$). These are subschemes of Y, and since Y is smooth, Proposition 3.36 tells us that the preimages $\pi_Y^{-1}(E_j)$ will also be irreducible.

We are only interested in the components whose preimages under π_Y contain at least one arc mapping its generic point outside of $f^{-1}(\operatorname{Sing} X)$. This choice is analogous to what we did in §4.2 with transversal arcs and Nash components. So set

$$\pi_Y^{-1}(E_j)^\circ := \pi_Y^{-1}(E_j) \setminus (f^{-1}(\operatorname{Sing} X))_\infty = \left\{ \begin{array}{c} \text{arcs centered on } E_j \text{ whose} \\ \text{generic point does not hit } f^{-1}(\operatorname{Sing} X) \end{array} \right\},$$

which will either be empty, or be a dense open subset of $\pi_Y^{-1}(E_j)$ for each $j = 1, \ldots, m$. The dense property comes from the fact that every $\pi_Y^{-1}(E_j)$ is irreducible.

Now we turn our attention to the singular locus of X. Let $\{C_i\}_{i \in \mathcal{I}}$ be the Nash components with respect to X and set

$$C_i^\circ := C_i \setminus (\operatorname{Sing} X)_\infty = \left\{ \begin{array}{c} \text{arcs centered on Sing X whose} \\ \text{generic point does not hit Sing X} \end{array} \right\}. \tag{4.1}$$

This time we have already ensured that C_i° will be nonempty, but just like above it will be a dense open subset of C_i for each $i \in \mathcal{I}$.

Consider the induced (cf. Proposition 3.16) canonical morphism $f_\infty : Y_\infty \to X_\infty$ between the arc spaces, which makes the following diagram commutative:

$$\begin{tikzcd} Y_\infty \arrow[r, "f_\infty"] \arrow[d, "\pi_Y"'] & X_\infty \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{4.2}$$

Since f is a proper morphism, we can apply Proposition 3.26 from which it follows that the restriction

$$f'_\infty : \mathrm{Y}_\infty \setminus (f^{-1}(\mathrm{Sing}\,\mathrm{X}))_\infty \to \mathrm{X}_\infty \setminus (\mathrm{Sing}\,\mathrm{X})_\infty$$

of f_∞, is a bijection. Then observe that, by virtue of the commutativity in (4.2),

$$\pi_\mathrm{Y}^{-1}(f^{-1}(\mathrm{Sing}\,\mathrm{X})) = f_\infty^{-1}(\pi_\mathrm{X}^{-1}(\mathrm{Sing}\,\mathrm{X}))$$

and so

$$\begin{aligned}\pi_\mathrm{Y}^{-1}(f^{-1}(\mathrm{Sing}\,\mathrm{X})) \setminus (f^{-1}(\mathrm{Sing}\,\mathrm{X}))_\infty &= f_\infty^{-1}(\pi_\mathrm{X}^{-1}(\mathrm{Sing}\,\mathrm{X})) \setminus (f^{-1}(\mathrm{Sing}\,\mathrm{X}))_\infty \\ &= (f'_\infty)^{-1}(\pi_\mathrm{X}^{-1}(\mathrm{Sing}\,\mathrm{X}) \setminus (\mathrm{Sing}\,\mathrm{X})_\infty).\end{aligned} \tag{4.3}$$

Noting that

$$\bigcup_{j=1}^{m} \pi_\mathrm{Y}^{-1}(\mathrm{E}_j)^\circ = \pi_\mathrm{Y}^{-1}(f^{-1}(\mathrm{Sing}\,\mathrm{X})) \setminus (f^{-1}(\mathrm{Sing}\,\mathrm{X}))_\infty$$

and

$$\bigcup_{i\in\mathcal{J}} \mathrm{C}_i^\circ = \pi_\mathrm{X}^{-1}(\mathrm{Sing}\,\mathrm{X}) \setminus (\mathrm{Sing}\,\mathrm{X})_\infty,$$

we then conclude from (4.3) that

$$\bigcup_{j=1}^{m} \pi_\mathrm{Y}^{-1}(\mathrm{E}_j)^\circ = (f'_\infty)^{-1}\left(\bigcup_{i\in\mathcal{J}} \mathrm{C}_i^\circ\right).$$

It follows that the restriction of f'_∞ to this preimage,

$$f''_\infty : \bigcup_{j=1}^{m} \pi_\mathrm{Y}^{-1}(\mathrm{E}_j)^\circ \to \bigcup_{i\in\mathcal{J}} \mathrm{C}_i^\circ,$$

is also a bijection. Note that f''_∞ is still continuous, with respect to the induced subspace topologies.

The big picture of what we have done so far is now illustrated in the following commutative diagram:

$$
\begin{tikzcd}[column sep=small]
& \bigcup_{j=1}^{m} \pi_Y^{-1}(E_j)^\circ \arrow[d, equal] & & \bigcup_{i \in \mathcal{I}} C_i^\circ \arrow[d, equal] \\
& \pi_Y^{-1}(f^{-1}(\operatorname{Sing} X)) \setminus (f^{-1}(\operatorname{Sing} X))_\infty \arrow[rr, "f''_\infty", "\text{bijective}"'] \arrow[dl, hook] \arrow[dd, hook] & & \pi_X^{-1}(\operatorname{Sing} X) \setminus (\operatorname{Sing} X)_\infty \arrow[dl, hook] \arrow[dd, hook] \\
Y_\infty \setminus (f^{-1}(\operatorname{Sing} X))_\infty \arrow[rr, "f'_\infty", "\text{bijective}"'] \arrow[dd, hook] & & X_\infty \setminus (\operatorname{Sing} X)_\infty \arrow[dd, hook] & \\
& \pi_Y^{-1}(f^{-1}(\operatorname{Sing} X)) \arrow[rr] \arrow[dl, hook] \arrow[dd, hook] & & \pi_X^{-1}(\operatorname{Sing} X) \arrow[dl, hook] \arrow[dd, hook] \\
Y_\infty \arrow[rr, "f_\infty"] \arrow[dd, "\pi_Y"'] & & X_\infty \arrow[dd, "\pi_X"] & \\
& f^{-1}(\operatorname{Sing} X) \arrow[rr] \arrow[dl, hook] & & \operatorname{Sing} X \arrow[dl, hook] \\
Y \arrow[rr, "f"'] & & X &
\end{tikzcd}
$$

The rest of the construction is just a topological argument.

Claim. For every $i \in \mathcal{I}$, there exists a unique $1 \le j \le m$ such that

$$f''_\infty(\xi_j) = \eta_i,$$

where η_i (resp. ξ_j) is the generic point of C_i (resp. $\pi_X^{-1}(E_j)$).

Proof. Fix an $i \in \mathcal{I}$. Since f''_∞ is surjective, we know that the preimage $(f''_\infty)^{-1}(\{\eta_i\})$ is nonempty. Moreover, by injectivity we know that this preimage will consist of a single point in $\pi_X^{-1}(E_j)^\circ$ for some $1 \le j \le m$. Now, by continuity we have

$$\eta_i \in f''_\infty(\pi_X^{-1}(E_j)^\circ) = f''_\infty(\overline{\{\xi_j\}}) \subseteq \overline{f''_\infty(\xi_j)},$$

meaning that

$$C_i = \overline{f''_\infty(\xi_j)}^{\pi_X^{-1}(\operatorname{Sing} X)}.$$

This second closure is being taken in $\pi_X^{-1}(\operatorname{Sing} X)$, of which C_i is an irreducible component. It follows that $f''_\infty(\xi_j) = \eta_i$.

If $\xi_j \in \pi_X^{-1}(E_{j'})$ for some $1 \le j' \le m$ then $j' = j$ by construction, whence uniqueness. ∎

So what have we actually done here? Well, since the choice of $i \in \mathcal{J}$ was arbitrary, we have constructed a well-defined map

$$\mathcal{N} : \left\{ \begin{array}{c} \text{generic points of} \\ \text{the Nash components} \end{array} \right\} \to \left\{ \begin{array}{c} \text{generic points} \\ \text{of the } \pi_Y^{-1}(E_j) \end{array} \right\}.$$

By construction, the Nash components and the preimages $\pi_Y^{-1}(E_j)$ all have distinct generic points. So we can write the map more succintly as

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\} \to \left\{ \begin{array}{c} \text{irreducible components} \\ \text{of } f^{-1}(\operatorname{Sing} X) \end{array} \right\}. \tag{4.4}$$

This is basically the Nash map, but we would like to be able to speak of it regardless of the choice of resolution f. So there is a bit more work to do, and in the mean time we shall informally refer to (4.4) as the *proto Nash map*.

Remark 4.23. We are using the continuity of the map $f''_\infty : \bigcup_{j=1}^m \pi_Y^{-1}(E_j)^\circ \to \bigcup_{i\in\mathcal{J}} C_i^\circ$, but remember that the inverse map is not necessarily continuous (cf. Remark 3.27). In particular, there is no guarantee that the image of irreducible subsets under $(f''_\infty)^{-1}$ will be irreducible.

4.5 Essential divisors

Definition 4.24. Let $f : X \to Y$ be a birational morphism of varieties, where $V \subseteq Y$ is the maximal open subset such that f induces an isomorphism $f^{-1}(V) \cong V$ (cf. Remark 4.21). The *exceptional set of* f is the closed subset $f^{-1}(Y \setminus V)$ of X.

We shall use the following definition of divisors and prime divisors on a variety. In actuality the notion of a divisor is not as simple—what we describe is rather the *support* of a divisor. But for our purposes this will suffice. For further detail, see for example [Har77, §II.6] or [EGA IV$_4$, §21].

Definition 4.25. Let X be a variety.

(a) A *prime divisor on* X is a closed irreducible subvariety of codimension 1.

(b) A *divisor on* X is a union of finitely many prime divisors on X.

Definition 4.26. Let $f : X \to Y$ be a birational morphism of varieties. A prime divisor on X is said to be an *exceptional divisor of* f if its image under f is of codimension ≥ 2. The exceptional divisors of f will be contained in the exceptional set of f.

Definition 4.27. Let X be a variety and let

$$f : Y_1 \to X$$
$$g : Y_2 \to X$$

be two proper birational morphisms from normal varieties Y_1 and Y_2 respectively, such that $E \subseteq Y_1$ is an irreducible exceptional divisor of f.

Following the above discussion regarding rational maps, we see that the composition $g^{-1} \circ f : Y_1 \dashrightarrow Y_2$ is a well-defined birational map. By Zariski's main theorem (see below), it is defined on some nonempty open subset E° of E. Then the closure of $g^{-1} \circ f(E^\circ)$ in Y_2 is called the *center of* E *on* Y_2. The definition is illustrated in the following commutative diagram:

$$\begin{tikzcd}
E^\circ \arrow[r, hook] \arrow[dd, "g^{-1}\circ f"'] & E \arrow[r, hook] & Y_1 \arrow[dd, dashed, "g^{-1}\circ f"'] \arrow[dr, "f"] & \\
& & & X \\
g^{-1}\circ f(E^\circ) \arrow[r, hook] & \overline{g^{-1}\circ f(E^\circ)} \arrow[r, hook] & Y_2 \arrow[ur, "g"] &
\end{tikzcd}$$

We denote the center of E on Y_2 by $\mathrm{center}_{Y_2}(E) := \overline{g^{-1} \circ f(E^\circ)}$.

Remark 4.28. What we call the center of E on Y_2 is also known as the *birational transform of* E with respect to the birational map $Y_1 \dashrightarrow Y_2$. See [Kol07, Definition 2.1].

There is a possibly nontrivial claim embedded in the above definition. Having presented some context, the explanation should now go more smoothly than if it were to precede the definition. Here we go.

Remark 4.29. Enter the context of Definition 4.27. We wish to show that the birational map

$$g^{-1} \circ f : Y_1 \dashrightarrow Y_2$$

is defined on some nonempty open subset $E^\circ \subseteq E$. This is an application of Zariski's main theorem,[2] which for our purposes can be formulated as follows:

Zariski's main theorem. *Suppose that* $g : Y \to X$ *is a proper birational morphism of varieties and that* Y *is normal. Then there exists an open subset* $U \subseteq X$ *such that* f *induces an isomorphism* $g^{-1}(U) \cong U$. *Moreover, the complement of* U *(that is,* $X \setminus U$*) has codimension* ≥ 2.

(This is a simplified version of [Liu02, Corollary 4.4.3(b)].[3])

Then take the closure of the graph of the rational map $Y_1 \dashrightarrow Y_2$ and normalise it (if necessary). Call the result Y_3. We then obtain two proper birational morphisms $Y_3 \to Y_1$ and $Y_3 \to Y_2$ from the corresponding projections. Now, Zariski's main theorem tells us that the morphism $Y_3 \to Y_1$ is an isomorphism for some dense open subset $U \subseteq Y_1$, and its complement has codimension ≥ 2. Hence there exists some nonempty open $E^\circ \subseteq U$, since E is of codimension

[2]One of many incarnations, anyway.

[3]This book [Liu02] has a very nice section on Zariski's result and its derivatives.

1. Then since Y_3 was obtained by taking the graph of $g^{-1} \circ f$, it follows that $g^{-1} \circ f$ is defined on E°.

Definition 4.30. Consider the context of Definition 4.27 again. We say that E *appears on* Y_2 if the center of E on Y_2 is also a divisor. In that case, we identify E with $\mathrm{center}_{Y_2}(E)$ to form an equivalence class of divisors which we simply denote by E. Such an equivalence class is called an *exceptional divisor over* X.

On the basis of the above definition, we will be able to reformulate the proto Nash map (4.4) independently of any particular resolution. A little extra work is still required, though.

Definition 4.31. Let X be a variety.

(a) An exceptional divisor E over X is said to be an *essential divisor over* X if for every resolution $f : Y \to X$, the center of E on Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

(b) Given a resolution $f : Y \to X$, the center of an essential divisor over X is called an *essential component over* Y.

Remark 4.32. Necessarily, every essential component over Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

This terminology might seem convoluted, but the next result and its proof should clarify what is being described. First, a quick lemma.

Lemma 4.33. *Let* X *and* Y *be varieties and* $Z \subseteq X$ *a subset. A birational morphism* $f : X \to Y$ *induces a bijection*

$$\left\{ \begin{array}{c} \text{generic points of the} \\ \text{irreducible components of } Z \end{array} \right\} \cong \left\{ \begin{array}{c} \text{generic points of the} \\ \text{irreducible components of } f(Z) \end{array} \right\}.$$

Proof. See [Sta15, Tag 01RN]. □

Proposition 4.34. *Let* $f : Y \to X$ *be a resolution of singularities of a variety* X. *Then there is a bijection*

$$\{ \text{ essential divisors over } X \} \cong \{ \text{ essential components over } Y \}.$$

Proof. We shall use the map

$$\begin{aligned} \{ \text{ essential divisors over } X \} &\to \{ \text{ essential components over } Y \} \\ E &\mapsto \mathrm{center}_Y(E) \end{aligned}$$

that we outlined implicitly in the above definitions. The first thing to note is that this is surjective by definition, so all we have to verify is the injectivity of this map.

$$\begin{tikzcd}
 & C \arrow[r, hook] & Y \arrow[dd, "f"] \\
E \arrow[r, hook] & Y' \arrow[ur, "g"] \arrow[dr] & \\
 & & X \\
F \arrow[r, hook] & Y'' \arrow[uu, "h"] \arrow[ur] &
\end{tikzcd}$$

Let $C \subseteq f^{-1}(\operatorname{Sing} X)$ be an essential component over Y and consider the blow-up $g : Y' \to Y$ of Y along C.

There exists a unique prime divisor on Y' whose generic point is mapped by g to the generic point of C. To see this, first note that blow-ups always have exceptional divisors. Let E denote this expectional divisor. Taking the restriction of f to E, Lemma 4.33 then tells us that f induces a bijection between the generic points of the irreducible components of E and the generic points of the irreducible components of C. But C is irreducible, therefore E is also irreducible. We conclude that E is unique a prime divisor with the required property.

Now take a resolution $g : Y'' \to Y'$ of Y'. Consider an arbitrary essential divisor F over X such that $\mathrm{center}_Y(F) = C$. Then we can consider the divisor $F \subseteq Y''$ of Y'' (cf. Definition 4.31). Observe how the center of F is derived:

$$\begin{tikzcd}
Y' \arrow[r, dashed, "f^{-1}\circ\varphi"] \arrow[dr, "\varphi"] & Y \arrow[d, "f"] \\
Y'' \arrow[u, dashed, "\varphi^{-1}\circ(\varphi\circ h)"] \arrow[r, "\varphi\circ h"'] & X
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
Y' \arrow[r, dashed, "f^{-1}\circ\varphi"] & Y \arrow[d, "f"] \\
Y'' \arrow[u, dashed, "\varphi^{-1}\circ(\varphi\circ h)"] \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi\circ h"'] & X
\end{tikzcd}$$

(Here, $\varphi := f \circ g$.) On the left we obtain two rational maps just like in Definition 4.27. On the right we note that the rational map $\psi : Y'' \dashrightarrow Y$ is just the composition of the two rational maps in the left diagram:

$$\psi \overset{\text{D4.27}}{:=} f^{-1} \circ \varphi \circ h = f^{-1} \circ (\varphi \circ \varphi^{-1} \circ \varphi) \circ h.$$

Given this fact together with the definition of E, it follows that

$$\mathrm{center}_Y(F) = C \implies \eta \in \psi(F^\circ) \implies \mathrm{center}_{Y'}(F) = E,$$

and hence F and E are the same essential divisor over X (cf. Definition 4.30). So by definition, any exceptional divisor with center C on Y will have its center on Y'' contained in E and hence this exceptional divisor will be identified with E. It follows that E is the only essential divisor over X with center C on Y. This establishes the injectivity of our map. $\square$

4.6 Statement of the problem

Observe that by definition,

$$\left\{ \text{ essential components over Y } \right\} \subseteq \left\{ \text{ irreducible components of } f^{-1}(\operatorname{Sing} \mathrm{X}) \right\}.$$

Proposition 4.35. *The image of the proto Nash map (4.4) consists only of essential components over* Y*, i.e.*

$$\operatorname{Im}(\mathcal{N}) \subseteq \left\{ \text{ essential components over Y } \right\}.$$

Proof. Fix a resolution $f : \mathrm{Y} \to \mathrm{X}$ and pick some Nash component C with respect to X. Let $\mathrm{E} \subseteq \mathrm{Y}$ be the irreducible component of $f^{-1}(\operatorname{Sing}\mathrm{X})$ corresponding to C, i.e. $\mathcal{N}(\mathrm{C}) = \mathrm{E}$. We wish to show that E is an essential component over Y, which means that it is the image of an essential divisor over X.

Pick up another resolution $\mathrm{Y}' \to \mathrm{X}$ and then let $\tilde{f} : \tilde{\mathrm{Y}} \to \mathrm{X}$ be a resolution of singularities that factors through both Y and Y′.

$$\begin{tikzcd}
 & \mathrm{Y} \arrow[dr, "f"] & \\
\tilde{\mathrm{Y}} \arrow[ur, "g"] \arrow[rr] \arrow[dr] & & \mathrm{X} \\
 & \mathrm{Y}' \arrow[ur] &
\end{tikzcd}$$

Accordingly, let $\mathrm{E}' \subseteq \mathrm{Y}'$ and $\tilde{\mathrm{E}} \subseteq \tilde{\mathrm{Y}}$ be the irreducible components corresponding to C for the respective Nash maps $\tilde{\mathcal{N}}$ and $\mathcal{N}'$.

First observe that $\mathrm{center}_{\mathrm{Y}}(\tilde{\mathrm{E}}) = \mathrm{E}$. To see this, let η be the generic point of C, and let ξ (resp. $\tilde{\xi}$) be the generic point of $\pi_{\mathrm{Y}}^{-1}(\mathrm{E})$ (resp. $\pi_{\tilde{\mathrm{Y}}}^{-1}(\tilde{\mathrm{E}})$). Then by definition, we have that

$$\begin{aligned}
\mathcal{N}(\mathrm{C}) = \mathrm{E} &\iff \eta = f_\infty''(\xi) = f \circ \xi \\
\tilde{\mathcal{N}}(\mathrm{C}) = \tilde{\mathrm{E}} &\iff \eta = (f \circ g)_\infty''(\tilde{\xi}) = f \circ g \circ \tilde{\xi},
\end{aligned}$$

hence $g \circ \tilde{\xi} = \xi$. Moreover we note that by continuity, $\xi(0)$ (resp. $\tilde{\xi}(0)$) is the generic point of E (resp. $\tilde{\mathrm{E}}$), and so $\xi(0) = g \circ \tilde{\xi}(0)$. Then since g sends the generic point of $\tilde{\mathrm{E}}$ to the generic point of E, it follows that $\mathrm{center}_{\mathrm{Y}}(\tilde{\mathrm{E}}) = \mathrm{E}$. So E is the center of the exceptional divisor $\tilde{\mathrm{E}}$ over X, and we identify E and $\tilde{\mathrm{E}}$ as exceptional divisors over X.

What remains to be seen is that E is an essential divisor over X, which would also make it an essential component over Y. This requires us to show that $\mathrm{center}_{\mathrm{Y}'}(\mathrm{E}) = \mathrm{center}_{\mathrm{Y}'}(\tilde{\mathrm{E}}) = \mathrm{E}'$. But this is easily seen by an argument identical to the one used just above, by virtue of E, E′ and $\tilde{\mathrm{E}}$ all corresponding to the same Nash component C. Therefore E is an essential component over Y. □

Definition 4.36. On this basis of the two results Proposition 4.34 and Proposition 4.35, we say that the map

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\} \to \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over X} \end{array} \right\} \tag{4.5}$$

is the *Nash map.*

Lemma 4.37 (Nash [Nas95, Proposition 2]). *The nash map* $\mathcal{N}$ *is injective.*

Proof. Pick an arbitrary resolution $f : Y \to X$. Let C and C' be two different Nash components with respect to X and suppose that $\mathcal{N}(C) = E = \mathcal{N}(C')$. Denote by η (resp. η') the generic point of C (resp. C'), and denote by ξ the generic point of $\pi_Y^{-1}(E)$. Then

$$\eta = f''_\infty(\xi) = \eta',$$

which means that $C = C'$. Contradiction ※. □

It is then natural to ask if this map is surjective (i.e. bijective). This is the Nash problem.

Problem 4.38 (Nash [Nas95, p. 36]). *Is the Nash map* $\mathcal{N}$ *bijective?*

Chapter 5

A proof for toric varieties

This chapter offers an accessible and rather elegant positive solution to the Nash problem in the case of toric varieties. The process is twofold: bijectivity of the Nash map is first proved for *affine* toric varieties, and then a final proof brings things together to solve the case in full generality for all toric varieties. The technique and proof of the affine case follows the paper [IK03] in which the result was first proven. The proof for general toric varieties is new.

This is a particularly nice demonstration because it works for toric varieties of any dimension and over an algebraically closed field of arbitrary characteristic. In this sense it is quite a strong result.

> *Assumptions.* Unless otherwise stated, k is an algebraically closed field of arbitrary characteristic.

5.1 Toric preliminaries

This chapter will be using a lot of stuff from the theory of toric varieties. Our standard reference here is [CLS11], which will be cited when necessary. The following definition recalls the main ideas that are needed, along with the corresponding notation.

Definition 5.1. (a) M is the free abelian group $\mathbb{Z}^n$ for $n \geq 2$.

(b) $\mathrm{N} := \mathrm{M}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ is the dual of M.

(c) $\mathrm{M} := \mathrm{M} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathrm{N} := \mathrm{N} \otimes_{\mathbb{Z}} \mathbb{R}$.

(d) A cone $\tau \subseteq \mathrm{N}$ is *smooth* if its generators form part of a $\mathbb{Z}$-basis of N. A fan Σ in $\mathrm{N}_{\mathbb{R}}$ is smooth if all of its constituent cones are smooth. If a cone (resp. fan) is not smooth, it is said to be *singular*.

(e) Accordingly we denote by $\langle -,- \rangle : N \times M \to \mathbb{Z}$ the natural paring between N and M. It extends to a pairing $\langle -,- \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$ between $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$.

(f) X_{Σ} denotes the toric variety of a given fan Σ in $N_{\mathbb{R}}$.

(g) $\operatorname{cone}(e_1, \ldots, e_r)$ is the cone in $N_{\mathbb{R}}$ generated by $e_1, \ldots, e_r \in N$.

(h) We use multi-index notation to write $x^u := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ for a given $m = (m_1, \ldots, m_n) \in M$. Accordingly, we write $k[M] := k[\{x^m\}_{m \in M}]$.

(i) $T := \operatorname{Spec} k[M]$ is the open orbit of a toric variety, i.e. the *embedded torus.*

(j) $\operatorname{orb}(\tau)$ denotes the torus orbit corresponding to a cone τ in $N_{\mathbb{R}}$.

(k) For a cone $\tau \in \Sigma$, U_{τ} denotes the affine toric variety of the cone τ, and is an invariant affine open subset of X_{Σ}.

(l) $D_v := \overline{\operatorname{orb}(\rho)}$ is the invariant prime divisor on a toric variety X_{Σ} for a given $v \in N$ generating a ray $\rho \in \Sigma$. By the orbit-cone correspondence [CLS11, Theorem 3.2.6], the D_v are precisely the irreducible components of $X \setminus T$.

5.2 Divisorial resolutions and divisorially essential divisors

We now add extra assumptions to a few previous definitions. During the proof of the affine case, we will then see that these definitions coincide in the case of toric varieties.

Definition 5.2. A scheme X is said to be of *pure dimension n* if all irreducible components of X have dimension n. For a closed subscheme $Z \subseteq X$ we say analogously that Z is of *pure codimension m* if all irreducible components of Z have codimension m.

The following definitions all have more general analogues that were presented in §4.5.

Definition 5.3 (cf. D4.22). A resolution of singularities $f : Y \to X$ is said to be a *divisorial resolution of singularities of* X if the exceptional set $f^{-1}(\operatorname{Sing} X)$ is of pure codimension 1.

Remark 5.4. If $f : Y \to X$ is a resolution of singularities such that Y is normal, then f is a divisorial resolution. This is a direct consequence of Zariski's main theorem; see [Mum99, Proposition III.9.1]. So in particular, we can always obtain a divisorial resolution from an existing resolution by composing with a normalization $Y' \to Y$.

Definition 5.5 (cf. D4.30 and D4.31). Let X be a variety.

(a) An exceptional divisor E over X is said to be a *divisorially essential divisor over* X if for every divisorial resolution $f : Y \to X$, E appears on Y, i.e. $\mathrm{center}_Y(E)$ is a divisor. Additionally, E is then an irreducible component of $f^{-1}(\mathrm{Sing}\, X)$.

(b) Given a divisorial resolution $f : Y \to X$, the center of a divisorially essential divisor over X is called a *divisorially essential component over* Y.

Remark 5.6. Given an arbitrary variety X, we have the following overview:

$$\left\{ \text{ divisorial resolutions of X } \right\} \subseteq \left\{ \text{ resolutions of X } \right\}$$
$$\left\{ \text{ divisorially essential divisors over X } \right\} \supseteq \left\{ \text{ essential divisors over X } \right\}$$

Warning 5.7. Note the direction of these inclusions! Since we conclude the latter from the former, the reversing of inclusions is to be expected.

Whether the latter inclusion is actually an equality seems not to have been established in general; see [IK03]. However in the case of toric varieties, we *do* have equality—more on this later.

Definition 5.8. An exeptional divisor E over a toric variety X is said to be a *toric divisorially essential divisor over* X if it appears on every equivariant divisorial resolution of singularities of X.

Remark 5.9. If X is a toric variety, then we have

$$\left\{ \begin{matrix} \text{toric divisorially} \\ \text{essential divisors over X} \end{matrix} \right\} \supseteq \left\{ \begin{matrix} \text{divisorially essential} \\ \text{divisors over X} \end{matrix} \right\} \supseteq \left\{ \begin{matrix} \text{essential divisors} \\ \text{over X} \end{matrix} \right\}.$$

5.3 Basic machinery

To begin with we shall consider an arbitrary *affine* toric variety, along with a few other fixed objects. They go as follows:

Definition 5.10. (a) $\sigma := \mathrm{cone}(e_1, \ldots, e_s) \subseteq N_{\mathbb{R}}$ is a cone generated by some primitive vectors $e_1, \ldots, e_s \in N_{\mathbb{R}}$ $(s \in \mathbb{N})$.

(b) Σ is the fan in $N_{\mathbb{R}}$ consisting of all faces of a cone $\sigma \subseteq N_{\mathbb{R}}$.

(c) $X := X_\Sigma$ is the affine toric variety of the fan Σ.

(d) T is the open orbit in X, i.e. the embedded torus.

(e) $W := \mathrm{Sing}\, X = \bigcup_{\substack{\tau \in \Sigma \\ \tau \text{ singular}}} \mathrm{orb}(\tau)$ is the singular locus of X. The latter equality comes from [CLS11, Proposition 11.1.2].

(f) $S := N \cap \bigcup_{\substack{\tau \in \Sigma \\ \tau \text{ singular}}} \tau^\circ$, where τ° denotes the relative interior of a cone $\tau \subseteq N_{\mathbb{R}}$.

Remark 5.11. Note that $T \in X \setminus W$. To see this, observe that $T = \mathrm{orb}(\{0\})$, where $\{0\}$ is the cone consisting of one point (the origin $0 \in N$). The cone $\{0\}$ is clearly nonsingular, and hence $T \cap W = \emptyset$.

Proposition 5.12. *If D_v is a toric divisorially essential divisor over X for some $v \in N \cap \sigma$ generating a ray $\rho \in N$, then $v \in S$.*

Proof. Let $f : Y \to X$ be an arbitrary equivariant divisorial resolution of singularities of X. The image of D_v under f will lie in the singular locus W,

$$f(D_v) = f(\overline{\mathrm{orb}(\rho)}) \subseteq \bigcup_{\substack{\tau \in \Sigma \\ \tau \text{ singular}}} \mathrm{orb}(\tau),$$

from which it follows that $v \in \bigcup_{\substack{\tau \in \Sigma \\ \tau \text{ singular}}} \tau^\circ$ by equivariance of f. By the hypothesis, ρ is in N as well, so we conclude that $\rho \in S$. □

Definition 5.13. (a) For every $u, v \in N \cap \sigma$, we define the relation

$$u \leq v \iff v \in \sigma + u.$$

(b) Let $A \subseteq N \cap \sigma$. An element $a \in A$ is said to be *minimal in* A if there exists no other element $a' \in A$ such that $a' \leq a$.

Proposition 5.14. *(a) The relation $\leq$ given in Definition 5.13 defines a partial order on $N \cap \sigma$.*

(b) For any $u, v \in N \cap \sigma$,

$$u \leq v \iff \forall\, m \in M \cap \sigma^\vee \;:\; \langle u, m\rangle \leq \langle v, m\rangle.$$

Proof. (a) Omitted.

(b)

$$\begin{aligned} u \leq v &\iff 0 \leq v - u \iff v - u \in \sigma \\ &\iff \forall\, m \in M \cap \sigma^\vee \;:\; 0 \leq \langle v - u, m\rangle \\ &\iff \forall\, m \in M \cap \sigma^\vee \;:\; \langle u, m\rangle \leq \langle v, m\rangle. \end{aligned}$$

□

Definition 5.15. Given an arc $\alpha : \operatorname{Spec} K[[t]] \to X$ such that $\alpha(\eta) \in T$, there is an associated element $u_\alpha \in N \cap \sigma$ defined as follows:

$$\begin{tikzcd} \operatorname{Spec} K((t)) \arrow[r, "\alpha"] \arrow[d, hook] & T \arrow[d, hook] \\ \operatorname{Spec} K[[t]] \arrow[r, "\alpha"'] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K((t)) & k[M] \arrow[l, "\alpha^*"'] \\ K[[t]] \arrow[u, hook] & k[M \cap \sigma^\vee] \arrow[l, "\alpha^*"] \arrow[u, hook] \end{tikzcd} \tag{5.1}$$

Consider the group homomorphism

$$\begin{aligned} M &\to \mathbb{Z} \\ m &\mapsto \operatorname{ord}(\alpha^* x^m), \end{aligned}$$

which uniquely determines an element $u_\alpha \in N$ such that

$$\langle u_\alpha, m \rangle = \operatorname{ord}(\alpha^* x^m) \;\; \forall\, m \in M.$$

(Here, $\operatorname{ord}(f)$ is the *order* of the formal Laurent series $f \in K((t))$, i.e. the smallest power of t with nonzero coefficient.) Moreover, by the commutative diagram of ring homomorphisms in (5.1) we see that

$$\langle u_\alpha, m \rangle \geq 0 \;\; \forall\, m \in M \cap \sigma^\vee,$$

and hence $u_\alpha \in N \cap \sigma$.

Warning 5.16. The above definition introduces u_α as a piece of formal notation, which is derived from a suitable arc α. Unless otherwise stated,[1] u_α is not derived from an element $u \in N$.

Proposition 5.17. *Consider an arc* $\alpha \in X_\infty$ *such that* $\alpha(\eta) \in T$, *and let* $\tau \preceq \sigma$ *be a face of* σ. *Then*

$$\alpha(0) \in \operatorname{orb}(\tau) \iff u_\alpha \in \tau^\circ.$$

In particular, $\alpha(0) \in T \iff u_\alpha = 0$.

Proof. Fix a face $\tau \preceq \sigma$. First note that

$$\operatorname{orb}(\tau) = U_\tau \setminus \bigcup_{\tau' < \tau} U_{\tau'}$$

and

$$\tau^\circ = \tau \setminus \bigcup_{\tau' < \tau} \tau',$$

by the orbit-cone correspondence [CLS11, Theorem 3.2.6]. Therefore to prove the claim, it suffices to show that

$$\alpha(0) \in U_\tau \iff u_\alpha \in \tau.$$

[1] And trust me, I will avoid this.

Then we have

$$\begin{aligned} u_\alpha \in \tau &\iff \forall\, m \in M \cap \tau^\vee \ :\ \langle u_\alpha, m\rangle \geq 0 \\ &\iff \alpha^* : k[M \cap \sigma^\vee] \to K[[t]] \text{ extends to } \alpha^* : k[M \cap \tau^\vee] \to K[[t]] \\ &\iff \alpha : \operatorname{Spec} K[[t]] \to X \text{ factors through } U_\tau \\ &\iff \alpha(0) \in U_\tau. \end{aligned}$$ □

Corollary 5.18. *Consider a subdivision Σ' of the fan Σ. Let $Y := Y_{\Sigma'}$ be the toric variety of Σ' and let $f : Y \to X$ be the corresponding toric morphism. Then given an arc $\alpha \in X_\infty$ such that $\alpha(\eta) \in T$, it follows that*

(a) the arc α is lifted to a unique arc $\tilde{\alpha} \in Y_\infty$,

(b) $u_\alpha = u_{\tilde{\alpha}}$, and

(c) for all cones $\tau \in \Sigma$,

$$\tilde{\alpha}(0) \in \operatorname{orb}(\tau) \iff u_\alpha = u_{\tilde{\alpha}} \in \tau^\circ.$$

Proof. (a) The unique lifting

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ \operatorname{Spec} K[[t]] \arrow[ur, dotted, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

is an immediate consequence of Proposition 3.26 and the fact that the induced toric morphism of a subdivision is always proper and induces an isomorphism on T; see [CLS11, §3].

(b) As $\tilde{\alpha}(\eta) \in T$ as well, and toric morphisms are equivariant, it follows that the corresponding ring homomorphisms of α and $\tilde{\alpha}$ are equal, i.e. $\alpha^* = \tilde{\alpha}^* : k[M] \to K((t))$. Then they define the same group homomorphism as in Definition 5.15, so also the same element $u_\alpha = u_{\tilde{\alpha}} \in N$.

(c) If we replace X with Y in the statement of Proposition 5.17 then this claim follows immediately, since $\Sigma \subseteq \Sigma'$. □

Proposition 5.19. *For every element $v \in S$, there exists an arc $\alpha : \operatorname{Spec} k[[t]] \to X$ such that $\alpha(\eta) \in T$, $\alpha(0) \in W$, and $u_\alpha = v$.*

Proof. Consider the ring homomorphism

$$\begin{aligned} \alpha^* : k[M] &\to k((t)) \\ x^m &\mapsto t^{\langle v, m\rangle}. \end{aligned}$$

Since $v \in S \subseteq \sigma$, it follows that $\langle v, m\rangle \geq 0$ for all $m \in M \cap \sigma^\vee$. So we can write a commutative diagram as follows:

$$\begin{tikzcd} k((t)) & k[M] \arrow[l, "\alpha^*"'] \\ k[[t]] \arrow[u, hook] & k[M \cap \sigma^\vee] \arrow[l, "\alpha^*"] \arrow[u, hook] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \operatorname{Spec} k((t)) \arrow[r, "\alpha"] \arrow[d, hook] & T \arrow[d, hook] \\ \operatorname{Spec} k[[t]] \arrow[r, "\alpha"'] & X \end{tikzcd} \tag{5.2}$$

Then choose the morphism $\alpha : \operatorname{Spec} k[[t]] \to X$ corresponding to α^*. For all $m \in M$,

$$\operatorname{ord}(\alpha^* x^m) = \operatorname{ord}(x^{\langle v,m\rangle}) = \langle v, m\rangle$$

and hence $v = u_\alpha$.

From the commutative diagram of scheme morphisms in (5.2), it is clear that $\alpha(0) \in T$. To see that $\alpha(\eta) \in W$, observe that since $v \in S$, there exists a singular face $\tau \prec \sigma$ such that $u_\alpha = v \in \tau^\circ$. Hence $\alpha(\eta) \overset{\text{P5.17}}{\in} \operatorname{orb}(\tau) \subseteq W$. □

Proposition 5.20 (Upper semi-continuity). *Let Z be a scheme over k and consider a family of arcs (cf. Definition 3.12)*

$$\alpha : Z \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X,$$

where each point $z \in Z$ corresponds to an arc $\alpha_z : \operatorname{Spec} k(z)[[t]] \to X$. If $\alpha_z(\eta) \in T$ for all $z \in Z$, then the map

$$\begin{aligned} Z &\to N \cap \sigma \\ z &\mapsto u_{\alpha_z} \end{aligned}$$

is upper semi-continuous, *meaning that for each $v \in N \cap \sigma$, the subset*

$$U_v := \{z \in Z \mid u_{\alpha_z} \leq v\} \subseteq Z$$

is open in Z.

Proof. Firstly, observe that the hypothesis $\alpha_z(\eta) \in T$ is necessary if the element u_{α_z} is to be well-defined (cf. Definition 5.15).

It suffices to verify the affine case, so suppose that $Z = \operatorname{Spec} A$ for some k-algebra A. We have a corresponding ring homomorphism

$$\alpha^* : k[M \cap \sigma^\vee] \to A[[t]]$$

with a mapping which we shall write as

$$x^m \mapsto a_{m,0} + a_{m,1} t + a_{m,2} t^2 + \cdots,$$

for suitable $a_{mi} \in \mathrm{A}$ where $i \in \mathbb{N}_0$. Then given a point $z \in \mathrm{Z}$, the arc α_z corresponds to a ring homomorphism

$$\begin{aligned} \alpha_z^* : k[\mathrm{M} \cap \sigma^\vee] &\to \mathrm{A}[[t]] \to k(z)[[t]] \\ x^m &\mapsto z_{m,0} + z_{m,1}t + z_{m,2}t^2 + \cdots , \end{aligned}$$

for $z_{m,i} := i_z^*(a_{m,i}) \in k(z)$ where $i \in \mathbb{N}_0$. (Here, $i_z^* : \mathrm{A} \to k(z)$ is the ring homomorphism induced by the canonical morphism $i_z : \operatorname{Spec} k(z) \to \mathrm{Z}$; see the discussion following Definition 3.12.)

Fix an arbitrary element $v \in \mathrm{N} \cap \sigma$ and note that for any $z \in \mathrm{Z}$,

$$z \in \mathrm{U}_v \iff u_{\alpha_z} \leq v \overset{\text{P5.14}}{\iff} \forall\, m \in \mathrm{M} \cap \sigma^\vee \;:\; \langle u_{\alpha_z}, m\rangle \leq \langle v, m\rangle. \tag{5.3}$$

The trick then goes as follows: $\mathrm{M} \cap \sigma^\vee$ is by definition generated by some *finite* set of elements $\{m_i\}_{i=1}^r \subseteq \mathrm{M} \cap \sigma^\vee$ ($r \in \mathbb{N}$), and therefore the condition in (5.3) is equivalent to

$$\forall\, j = 1, \ldots, r \;:\; \exists\, \ell \leq \langle v, m_j\rangle \text{ such that } z_{m_j,\ell} \neq 0.$$

Now it's actually better to think instead of when $z \notin \mathrm{U}_v$:

$$z \notin \mathrm{U}_v \iff \forall \left\{ \begin{array}{l} j = 1, \ldots, r \\ \ell = 1, \ldots, \langle v, m_j\rangle \end{array} \right. \;:\; z_{m_j,\ell} = 0.$$

Since $z_{m_j,\ell} = i_z^*(a_{m_j,\ell})$, we can then write the complement of U_v as a zero locus as follows:

$$\begin{aligned} \mathrm{Z} \setminus \mathrm{U}_v &= \left\{ a_{m_j,\ell} = 0 \;:\; \begin{array}{c} j = 1, \ldots, r \\ \ell = 1, \ldots, \langle v, m_j\rangle \end{array} \right\} \\ &= \left\{ \sum_{\ell=1}^{\langle v,m_j\rangle} a_{m_j,\ell} = 0 \;:\; j = 1, \ldots, r \right\} \\ &= \bigcap_{j=1}^{r} \left\{ \sum_{\ell=1}^{\langle v,m_j\rangle} a_{m_j,\ell} = 0 \right\}. \end{aligned}$$

This complement is then closed in Z with respect to the Zariski topology, hence U_v is open in Z. Our choice of $v \in \mathrm{N} \cap \sigma$ was abitrary, whence the result. □

Corollary 5.21. *Let* $\alpha : \mathrm{Z} \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to \mathrm{X}$ *be a family of arcs for some* $\mathrm{Z} \in \mathcal{S}\mathrm{ch}/k$. *Suppose that* $\alpha_z(\eta) \in \mathrm{T}$ *for all* $z \in \mathrm{Z}$. *If* u_{α_y} *is minimal in* S *for some* $y \in \mathrm{Z}$, *then the subset*

$$\{z \in \mathrm{Z} \mid u_{\alpha_z} = u_{\alpha_y}\} \subseteq \mathrm{Z}$$

is nonempty and open in Z.

Proof. This is immediate by the preceding result and the definition of minimality (Definition 5.13). □

Now let $\{C_i\}_{i\in\mathcal{I}}$ be the Nash components with respect to X (cf. Definition 4.9). By Corollary 3.13 there exists a universal family of arcs

$$\alpha : X_\infty \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X,$$

from which we obtain a corresponding family of arcs

$$\alpha_i : C_i \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X$$

for each $i \in \mathcal{I}$. In line with the notation used previously, the arc associated to each point $z \in C_i$ can be written $\alpha_{iz} : \operatorname{Spec} k(z)[[t]] \to X$. But this coincides with α_x, therefore we drop the additional i and just write

$$\alpha_z : \operatorname{Spec} k(z)[[t]] \to X.$$

The following result inherits this notation.

Corollary 5.22. *For each minimal element $v \in S$, there exists a Nash component C_i (for suitable $i \in \mathcal{I}$) and a nonempty open subset $U \subseteq C_i$ such that $u_{\alpha_z} = v$ for all $z \in U$.*

Proof. Fix an arbitrary minimal element $v \in S$.

By Proposition 5.19, there exists an arc $\beta : \operatorname{Spec} k[[t]] \to X$ such that $\beta(\eta) \in T$, $\beta(0) \in W$, and $u_\beta = v$. Since this arc has center on the singular locus of X, it must belong to some Nash component C_i for suitable $i \in \mathcal{I}$. In particular, we have that $\beta \in C_i^\circ$, where $C_i^\circ := C_i \setminus W$ (as in (4.1)). Recall that C_i° is open in C_i. Moreover, the arc β corresponds to a k-valued point $y \in C_i$ and so $\beta = \alpha_y$.

Now, observe that we have

$$\alpha_i(C_i \times_{\operatorname{Spec} k} \{0\}) \subseteq W$$

and $\alpha_y(\eta) \in T$. Therefore there exists an open neighbourhood $V \subseteq C_i$ of α_y such that

$$\alpha_i(V \times_{\operatorname{Spec} k} \{0\}) \subseteq W$$

(obviously), and that

$$\alpha_z(\eta) \in T$$

for all $z \in V$.

Then just apply Corollary 5.21 on the family of arcs

$$V \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \to X$$

to obtain a nonempty open subset

$$U := \{z \in V \mid u_{\alpha_z} = v\} \subseteq C_i.$$

□

5.4 Affine case

Now we are ready to prove bijectivity of the Nash map for affine toric varieties. The proof is rather more a collection of unassuming lemmas, followed by a paragraph and a commutative diagram. This is my favourite kind of proof.

Definition 5.23. For each minimal element $v \in S$, pick up an $i_v \in \mathcal{J}$ such that C_{i_v} is satisfied by Corollary 5.22. Then define the map

$$\mathcal{F} : \left\{ \text{ minimal elements in S } \right\} \to \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\}$$
$$v \mapsto C_{i_v}.$$

While this map is indeed well-defined, there is *a priori* no canonical choice of Nash component for each minimal element of S. What *is* immediately clear, however, is that $\mathcal{F}$ is an injective map.

Lemma 5.24. *The map $\mathcal{F}$ is injective.*

Proof. Suppose that $\mathcal{F}(v) = \mathcal{F}(w) = C_i$ for two minimal elements $v, w \in S$ and suitable $i \in \mathcal{J}$. Since every open subset of the irreducible component C_i contains the generic point, we have

$$\{z \in V_v \mid u_{\alpha_z} = v\} \cap \{z \in V_w \mid u_{\alpha_z} = w\} \neq \emptyset$$

for suitable $V_v, V_w \subseteq C_i$ like the V constructed in the proof of Corollary 5.22. This means that $v = w$. □

Now recall the Nash map defined in (4.5):

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\} \to \left\{ \text{ essential divisors over X } \right\}.$$

Lemma 5.25. *The composition $\mathcal{N} \circ \mathcal{F}$ behaves as follows:*

$$\mathcal{N} \circ \mathcal{F} : \left\{ \text{ minimal elements in S } \right\} \to \left\{ \text{ essential divisors over X } \right\}$$
$$v \mapsto D_v.$$

Proof. Fix an arbitrary minimal element $v \in S$.

By definition and by Corollary 5.22, the generic point of $\mathcal{F}(v)$ will correspond to an arc $\alpha : \operatorname{Spec} K[[t]] \to X$ such that $\alpha(\eta) \in T$ and

$$u_\alpha = v. \tag{5.4}$$

Then consider a toric divisorial resolution $f : Y \to X$ and let $\tilde{\alpha} \in Y_\infty$ be the unique lifting of α (cf. Corollary 5.18(a)).

Notice that the Nash map is constructed from the same lifting that we used to obtain our α. To see this, recall Proposition 3.26, which is the original result that led to the construction of $\mathcal{N}$ (§4.4) and the lifting of α (Corollary 5.18(a)).

So $\mathcal{N} \circ \mathcal{F}(v)$ will be an essential divisor over X, which corresponds to an essential component $E \subseteq f^{-1}(W)$ (cf. Proposition 4.35). Following the remarks just made about $\mathcal{N}$ and the lifting of α, the generic point of $\pi_Y^{-1}(E)$ is then $\tilde{\alpha}$. Lastly, by the continuity of π_Y and irreducibility of E we can see that the generic point of E is $\tilde{\alpha}(0)$.

By Definition 5.3, the essential component E is also a divisor on Y, and we shall denote it by $E = \overline{\mathrm{orb}(\rho)}$ for a suitable ray $\rho \subseteq N$ of the fan of Y. As $\tilde{\alpha}(0)$ is the generic point of $\overline{\mathrm{orb}(\rho)}$, it must be that $\tilde{\alpha}(0) \in \mathrm{orb}(\rho)$. Then by Corollary 5.18(b,c), we have that

$$v \overset{(5.4)}{=} u_\alpha = u_{\tilde{\alpha}} \in \rho^\circ.$$

Since ρ is 1-dimensional and v is a minimal, we conclude that $\overline{\mathrm{orb}(\rho)} = D_v$ and hence $\mathcal{N} \circ \mathcal{F}(v) = D_v$. □

Consider a map

$$\begin{aligned} \mathcal{G} : \left\{ \begin{matrix} \text{toric divisorially} \\ \text{essential divisors over X} \end{matrix} \right\} &\to S \\ D_v &\mapsto v. \end{aligned}$$

We are going to show that the image of $\mathcal{G}$ contains only minimal elements of S. First, a lemma.

Lemma 5.26. *Every element $v \in S$ that is not minimal can be written $v = n_1 + n_2$ for $n_1 \in S$ and such that n_2 satisfies either* (1) $n_2 \in S$, or (2) $n_2 \in \Sigma(1)$.

Proof. It's useful to recall the definition of the set S:

$$S = N \cap \bigcup_{\substack{\tau \in \Sigma \\ \tau \text{ singular}}} \tau^\circ. \tag{5.5}$$

Pick up an element $v \in S$ and suppose that v is *not* minimal. Then by Definition 5.13(a), $v \in \sigma + n_1$ for some $n_1 \in S$. Write

$$v = n_1 + n_2$$

for suitable $n_2 \in N \cap \sigma \setminus \{0\}$. We shall examine the two possible cases

(1) $n_2 \in S$,

(2) $n_2 \notin S$.

Let's first consider case (2)—that $n_2 \notin S$. Then by (5.5), n_2 must be contained in some nonsingular face $\tau \prec \sigma$. Without loss of generality, we write

$$\tau = \operatorname{cone}(e_1, \ldots, e_d)$$

for suitable primitive vectors $e_1, \ldots, e_d$ and $d < s$. Remember here that $\sigma = \operatorname{cone}(e_1, \ldots, e_s)$ (cf. Definition 5.10). Then

$$n_2 = \sum_{i=1}^{d} \lambda_i e_i$$

for suitable elements $\lambda_1, \ldots, \lambda_d \in \mathbb{N}_0$, not all of which are zero. Suppose further, and without loss of generality, that $\lambda_1 \neq 0$. Now let $\gamma \prec \sigma$ be the smallest face that contains the cone $\operatorname{cone}(n_1, \sum_{i=2}^{d} \lambda_i e_i)$ (note that we have removed the λ_1-term!). Clearly $n_1 \in \gamma$, and since $n_1 \in S$ it must be that γ is a singular face. Moreover, this means that

$$n_1 + \sum_{i=2}^{d} \lambda_i e_i \in \gamma^\circ \subseteq S.$$

So if we set

$$\begin{aligned} n_1' &:= n_1 + \sum_{i=2}^{d} \lambda_i e_i \in S \\ n_2' &:= \lambda_1 e_1 \in \Sigma(1), \end{aligned}$$

then $v = n_1' + n_2'$. By this process we can reformulate, without loss of generality, the second case to

(2) $n_2 \in \Sigma(1)$

by ensuring a suitable choice of n_1, n_2. In words, this case means that n_2 is a ray of the fan Σ and hence a 1-dimensional face of σ. □

Definition 5.27. (a) A cone is said to be *simplicial* if it has linearly independent generators over $\mathbb{R}$.

(b) Let $\tau \subseteq N_{\mathbb{R}}$ be a simplicial cone generated by $u_1, \ldots, u_r \in N$. The *multiplicity of* τ is the number of elements in $P_\tau \cap N$, where

$$P_\tau := \left\{ \sum_{i=1}^{r} \lambda_i u_i \;\middle|\; 0 \leq \lambda_i < 1 \right\}.$$

Lemma 5.28. *A cone* $\tau \subseteq N_{\mathbb{R}}$ *is smooth if, and only if,* $\operatorname{mult}(\tau) = 1$.

Proposition 5.29. *The image of the map* $\mathcal{G}$ *contains only minimal elements of* S.

Proof. Pick up an element $v \in S$ and suppose again that it is *not* minimal. By Lemma 5.26, we write

$$v = n_1 + n_2$$

for $n_1 \in S$ and such that either

(1) $n_2 \in S$, or

(2) $n_2 \in \Sigma(1)$.

Now, consider the 2-dimensional cone $\text{cone}(n_1, n_2)$ and let Δ denote the minimal smooth subdivison of that cone, which by [BGS95, Proposition 1.8] we know will always exist. Then v lies in a particular cone $\tau \in \Delta$, and we shall write $\tau = \text{cone}(v_1, v_2)$ for suitable primitive vectors $v_1, v_2 \in N \cap |\Delta|$. Since v is not minimal and Δ is the minimal subdivision of $\text{cone}(n_1, n_2)$, it must be that $v \in \tau^\circ$. One of the rays of τ will also have to be in the interior of a singular face of σ and hence in S, so assume without loss of generality that $v_1 \in S$.

The plan is to construct a smooth subdivision of Σ that contains τ. In this way, τ will "protect" the point v from becoming part of a ray of the subdivision, and from this we will conclude that D_v is not a toric divisorially essential divisor.

Take the star subdivision Σ_1 of Σ with center v_1. Next, depending on whether we are in case (1) or (2), set

$$\Sigma_2 := \begin{cases} \text{star subdivision of } \Sigma_1 \text{ with center } v_2 & (1), \\ \Sigma_1 & (2). \end{cases}$$

We want Σ_2 to be a simplicial subdivision, i.e. every cone in Σ is simplicial. If it's not, we make a simplicial fan in the following way: Pick up a non-simplicial minimal-dimensional cone $\gamma \in \Sigma_2$ and an element $n \in \gamma^\circ$. Then take the star subdivision of Σ_2 with center n, which splits γ into a union of simplicial cones. Keep doing this for every such non-simplicial minimal-dimensional cone until we are left with none, and call the result Σ_3.

We are almost done, but first we must account for the possibility that Σ_3 is a singular fan. The process by which we resolve this problem is similar to our approach regarding simlicial cones. Pick up a cone $\delta := \text{cone}(p_1, \ldots, p_d)$ with maximal multiplicity $\text{mult}(\delta)$ for suitable primitive vectors $p_1, \ldots, p_d$, $d \in \mathbb{N}$. Since Σ_3 is not smooth, we know that such a cone will have $\text{mult}(\delta) > 1$ by Lemma 5.28. Then there exists a nonzero element $n' \in P_\delta \cap N$, and we take the star subdivision of Σ_3 with center n'. Like before, repeat this process until the multiplicity of every cone is 1, meaning that our fan will be smooth. Call this smooth fan Σ_4.

Throughout this process we have not altered the cone $\text{cone}(n_1, n_2)$ at all, and so $\text{cone}(n_1, n_2) \in \Sigma_4$ as well. This means that the exceptional divisor D_v over X does *not* appear on the toric variety X_{Σ_4} corresponding to Σ_4. If in addition we find that the corresponding equivariant morphism $X_{\Sigma_4} \to X$ is a resolution of singularities, then the proof is complete.

But this is easy to notice, since we obtained the subdivision Σ_4 purely through a process of repeated star subdivisions inside singular cones. Hence the morphism $X_{\Sigma_4} \to X$ is an isomorphism outside the singular locus of X by [CLS11, Proposition 11.1.2]. Since Σ_4 is smooth by construction, it follows that we have a resolution of singularities. So we are done.

□

Definition 5.30. We therefore define the map $\mathcal{G}$ to formally be

$$\mathcal{G} : \left\{ \begin{matrix} \text{toric divisorially} \\ \text{essential divisors over X} \end{matrix} \right\} \to \left\{ \text{ minimal elements in S } \right\}$$
$$D_v \mapsto v.$$

The following comes for free by the definition of this map.

Lemma 5.31. *The map $\mathcal{G}$ is injective.*

The main lemma is now remarkably easy to show.

Lemma 5.32 (Ishii-Kollár [IK03, Theorem 3.16]). *If X is an affine toric variety, then the Nash map*

$$\mathcal{N} : \left\{ \begin{matrix} \text{Nash components} \\ \text{with respect to X} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{essential divisors} \\ \text{over X} \end{matrix} \right\}$$

is bijective.

Proof. The maps $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{N}$ are injective by Lemma 5.24, Lemma 5.31, and Lemma 4.37 respectively. Combining Lemma 5.25 and the definition of $\mathcal{G}$ Definition 5.30, it then follows that $\mathcal{G} \circ \mathcal{F} \circ \mathcal{N}$ is the identity map on the set of minimal elements in S. The situation can be seen in the following diagram:

$$\begin{tikzcd}
v \arrow[rrr, mapsto, "\mathcal{N}\circ\mathcal{F}", "\mathrm{L}5.25"'] & & & {} \arrow[ddd] \\
& \left\{ \begin{matrix} \text{minimal} \\ \text{elements in S} \end{matrix} \right\} \arrow[r, "\mathcal{F}", "\mathrm{L}5.24"'] & \left\{ \begin{matrix} \text{Nash components} \\ \text{with respect to X} \end{matrix} \right\} \arrow[d, "\mathrm{L}4.37"', "\mathcal{N}"] & \\
& \left\{ \begin{matrix} \text{toric divisorially} \\ \text{essential divisors} \\ \text{over X} \end{matrix} \right\} \arrow[u, "\mathcal{G}", "\mathrm{L}5.31"'] & \left\{ \begin{matrix} \text{essential divisors} \\ \text{over X} \end{matrix} \right\} \arrow[l, hook', "\mathrm{R}5.9"'] & \\
{} \arrow[uuu] & & & D_v \arrow[lll, mapsto, "\mathrm{P}5.29 + \mathrm{D}5.30"', "\mathcal{G}"]
\end{tikzcd}$$

All maps in the above diagram are therefore bijective, and in particular the Nash map $\mathcal{N}$ is bijective. □

Corollary 5.33. *For any toric variety* X*, we have a bijection*

$$\left\{ \begin{array}{c} \text{toric divisorially} \\ \text{essential divisors over X} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over X} \end{array} \right\}.$$

5.5 The proof

What follows now is a proof of bijectivity of the Nash map for all toric varieties. Just like in the affine case, we prove some harmless facts and conclude with a commutative diagram.

To the best of my knowledge, no proof of this fact has been given before.

Theorem 5.34. *If* X *is a toric variety, then the Nash map*

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\} \to \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over X} \end{array} \right\}$$

is bijective.

Proof. Let $X = X_\Sigma$ for some arbitrary fan Σ, not necessarily affine. Let D_v be an essential divisor over X, for a suitable $v \in N$ generating the ray $\rho := v\mathbb{R}_{\geq 0}$. Let $\sigma \in \Sigma$ be the maximal cone containing ρ and consider the corresponding affine toric variety $\tilde{X} := U_\sigma$. Note that $\tilde{X} \hookrightarrow X$ is an open immersion, and moreover that $\tilde{X}$ is singular since it contains $\mathrm{orb}(\rho)$. Lastly, let $f : Y \to X$ be an equivariant divisorial resolution of singularities.

Set $\tilde{Y} := f^{-1}(\tilde{X})$ and $\tilde{f} := f|_{\tilde{Y}} : \tilde{Y} \to \tilde{X}$. It is clear that $\tilde{f}$ is also an equivariant divisorial resolution of singularities of $\tilde{X}$. Moreover, we have that

$$\tilde{f}^{-1}(\operatorname{Sing}\tilde{X}) = f^{-1}(\operatorname{Sing}\tilde{X}) = f^{-1}(\operatorname{Sing}X) \cap \tilde{Y}.$$

Now let $\{E_1, \dots, E_m\}$ be the irreducible components of $f^{-1}(\operatorname{Sing}X)$. Suppose without loss of generality that

$$\begin{aligned} \tilde{E}_j := E_j \cap \tilde{Y} &\neq \emptyset \ \ \forall\, 1 \leq j \leq \tilde{m} \\ E_j \cap \tilde{Y} &= \emptyset \ \ \forall\, \tilde{m} < j \leq m \end{aligned}$$

for a suitable $\tilde{m} \in \mathbb{N}$.

Claim (1). The $\{\tilde{E}_1, \dots, \tilde{E}_{\tilde{m}}\}$ are precisely the irreducible components of $\tilde{f}^{-1}(\operatorname{Sing}\tilde{X})$.

Proof. Pick up a $j \in \mathbb{N}$ such that $1 \leq j \leq \tilde{m}$. Firstly note that all of the following subsets are endowed with the subspace topology in the obvious sense:

$$\begin{array}{ccccc} E_j & \hookrightarrow & f^{-1}(\operatorname{Sing}X) & \hookrightarrow & Y \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{E}_j & \hookrightarrow & \tilde{f}^{-1}(\operatorname{Sing}\tilde{X}) & \hookrightarrow & \tilde{Y} \end{array}$$

We shall use consequences of this fact without reference, so as not to complicate things. The proof then boils down to the following sequence of propositions.

(a) ($\tilde{E}_j$ is an open subset of E_j.) This follows from the fact that $\tilde{Y}$ is an open subset of Y, and hence $\tilde{E}_j = E_j \cap \tilde{Y}$ is open in E_j.

(b) ($\tilde{E}_j$ is a closed subset of $\tilde{f}^{-1}(\operatorname{Sing}\tilde{X})$.) This is true for the same reasons as (a), noting that E_j is a closed subset of Y.

(c) ($\tilde{E}_j$ is irreducible.) By (a), $\tilde{E}_j$ contains the generic point η of E_j. So if $\tilde{E}_j = F_1 \cup F_2$ for closed subsets F_1 and F_2, then without loss of generality we can suppose that $\eta \in F_1$. But then

$$\tilde{E}_j = \overline{\{\eta\}}^{\tilde{E}_j} \subseteq \overline{F_1}^{\tilde{E}_j} = F_1,$$

meaning that $F_1 = \tilde{E}_j$, whence irreducibility. In particular, η is also the unique generic point of $\tilde{E}_j$.

(d) ($\tilde{E}_j$ is a maximal irreducible subset of $\tilde{f}^{-1}(\operatorname{Sing}\tilde{X})$.) Suppose for contradiction that there exists another irreducible closed subset $\tilde{F} \subseteq \tilde{f}^{-1}(\operatorname{Sing}\tilde{X})$ such that $\tilde{E}_j \subseteq \tilde{F}$, and let ξ be the generic point of $\tilde{F}$. Then $\tilde{F} = F \cap \tilde{Y}$ for some suitable closed subset $F \subseteq f^{-1}(\operatorname{Sing} X)$, and $\xi \in F$. Then

$$\eta \in \tilde{E}_j \subseteq \tilde{F} = \overline{\{\xi\}}^{\tilde{F}} \subseteq \overline{\{\xi\}}^{F} \subseteq \overline{\{\xi\}}^{f^{-1}(\operatorname{Sing} X)} \subseteq \overline{F}^{f^{-1}(\operatorname{Sing} X)} = F,$$

and hence $E_j = \overline{\{\eta\}}^{f^{-1}(\operatorname{Sing} X)} \subseteq F$. This leaves us with two cases:

(i) ($\xi \in E_j$.) $\implies$ $\tilde{F} = \tilde{E}_j$. ※

(ii) ($\xi \notin E_j$.) $\implies$ $\xi \in E_\ell$ for some $\ell \neq i$, meaning that

$$E_j \subseteq \overline{\{\xi\}}^{f^{-1}(\operatorname{Sing} X)} \subseteq \overline{E_\ell}^{f^{-1}(\operatorname{Sing} X)} = E_\ell. \text{ ※}$$

Both cases lead to a contradiction, so we are done. The claim then follows. ∎

Claim (2). For each $1 \leq j \leq \tilde{m}$, $\pi_{\tilde{Y}}^{-1}(\tilde{E}_j)$ is irreducible and has the same generic point as $\pi_Y^{-1}(E_j)$.[2]

Proof. Let $1 \leq j \leq \tilde{m}$. Since $\tilde{Y}$ is smooth and $\tilde{E}_j$ is irreducible (by Claim (1)), we apply Proposition 3.33. This tells us that $\pi_{\tilde{Y}}^{-1}(\tilde{E}_j)$ is irreducible.

Next, since $\tilde{E}_j$ is an open subset of E_j, we see that $\pi_{\tilde{Y}}^{-1}(\tilde{E}_j)$ is an open subset of $\pi_Y^{-1}(E_j)$. Moreover, $\pi_{\tilde{Y}}^{-1}(\tilde{E}_j)$ will inherit the generic point of $\pi_Y^{-1}(E_j)$. ∎

[2]In fact, the work needed to prove the first part of Claim (2) has already been done in the construction of the Nash map (§4.4). It works here too because we have already shown in Claim (1) that the $\{\tilde{E}_j\}_{j=1}^{\tilde{m}}$ are precisely the irreducible components of the preimage of the singular locus of $\tilde{X}$.

For the next claim, we know from Corollary 3.25 that $(\tilde{X})_\infty \hookrightarrow X_\infty$ is an open immersion, and by Corollary 3.24 we know that $(\tilde{X})_\infty = \pi_X^{-1}(\tilde{X})$. Then $\pi_{\tilde{X}} = \pi_X\big|_{(\tilde{X})_\infty}$ and $\pi_{\tilde{X}}^{-1}(\operatorname{Sing}\tilde{X}) = \pi_X^{-1}(\operatorname{Sing} X) \cap (\tilde{X})_\infty$. Now let $\{C_i\}_{i\in\mathcal{J}}$ be the Nash components with respect to X. Similarly to above, suppose that

$$\begin{aligned} \tilde{C}_i := C_i \cap (\tilde{X})_\infty &\neq \emptyset \quad \forall\, i \in \tilde{\mathcal{J}} \\ C_i \cap (\tilde{X})_\infty &= \emptyset \quad \forall\, i \in \mathcal{J} \setminus \tilde{\mathcal{J}} \end{aligned}$$

for a suitable nonempty subset $\tilde{\mathcal{J}} \subseteq \mathcal{J}$.

Claim (3). The $\{\tilde{C}_i\}_{i\in\tilde{\mathcal{J}}}$ are precisely the Nash components with respect to $\tilde{X}$.

Proof. Just like in Claim (1), we have a situation of the following form:

$$\begin{array}{ccccc} C_i & \hookrightarrow & \pi_X^{-1}(\operatorname{Sing} X) & \hookrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{C}_i & \hookrightarrow & \pi_{\tilde{X}}^{-1}(\operatorname{Sing}\tilde{X}) & \hookrightarrow & \tilde{X} \end{array}$$

By an identical argument to the kind given in Claim (1), it follows that the $\{\tilde{C}_i\}_{i\in\tilde{I}}$ are irreducible components of $\pi_{U_\alpha}^{-1}(\operatorname{Sing}\tilde{X})$. *But*, they are not necessarily all of the irreducible components, since we have taken only Nash components with respect to X and there might be some other irreducible components of $\pi_X^{-1}(\operatorname{Sing} X)$ left over. It remains to show two things: that each $\tilde{C}_i$ is a Nash component with respect to $\tilde{X}$, and that every Nash component with respect to $\tilde{X}$ is of this form.

The first of these facts can be easily verified by recalling what was shown in (c) of Claim (1); namely, that $\tilde{C}_i$ and C_i will share the same generic point. Then in particular, these generic points will be arcs on both $\tilde{X}$ and X. Moreover, as arcs on X they will be transversal (cf. (4.1)), and since $\operatorname{Sing}\tilde{X} \subseteq \operatorname{Sing} X$, we know that they will be transversal as arcs on $\tilde{X}$. Hence all the $\tilde{C}_i$ are Nash components.

Conversely, any irreducible component C' of $\pi_X^{-1}(\operatorname{Sing} X)$ which is *not* a Nash component will contain no transversal arcs. Then clearly the corresponding irreducible component $\tilde{C}' := C' \cap (\tilde{X})_\infty$ of $\pi_{\tilde{X}}^{-1}(\operatorname{Sing}\tilde{X})$ will also contain no transversal arcs. This proves the claim. ∎

Claim (4). For each $i \in \tilde{\mathcal{J}}$, $\tilde{C}_i$ has the same generic point as C_i.

Proof. This was already justified in the proof of Claim (3). ∎

Now let's briefly recall how we constructed the Nash map for X (§4.4). We took the induced morphism

$$\begin{aligned} f_\infty : Y &\to X \\ \alpha &\overset{R3.17}{\mapsto} f \circ \alpha \end{aligned}$$

and restrcited it to obtain a bijective map

$$f''_\infty : \bigcup_{j=1}^{m} \pi_{\mathrm{Y}}^{-1}(\mathrm{E}_j)^\circ \to \bigcup_{i \in \mathcal{I}} \mathrm{C}_i^\circ.$$

Then we defined $\mathcal{N}(\mathrm{C}_i) := \mathrm{E}_j$, where $j \in \mathbb{N}$ is the unique $1 \leq j \leq m$ such that f''_∞ maps the generic point of $\pi_{\mathrm{Y}}^{-1}(\mathrm{E}_j)$ to the generic point of C_i.

Claim (5). The bijective map

$$\tilde{f}''_\infty : \bigcup_{j=1}^{\tilde{m}} \pi_{\tilde{\mathrm{Y}}}^{-1}(\mathrm{E}_j)^\circ \to \bigcup_{i \in \tilde{\mathcal{I}}} \tilde{\mathrm{C}}_i^\circ$$

agrees with f''_∞ on the domain of $\tilde{f}''_\infty$.

Proof. Just observe that $\tilde{f}_\infty = \tilde{f} \circ - = f \circ -$, and so $\tilde{f}''_\infty = f \circ -$. Yes, it's that easy! ■

Now suppose without loss of generality that

$$\{\tilde{\mathrm{E}}_1, \dots, \tilde{\mathrm{E}}_{\tilde{m}'}\} = \{ \text{ essential components over } \tilde{\mathrm{Y}} \ \} \tag{5.6}$$

$$\{\mathrm{E}_1, \dots, \mathrm{E}_{m'}\} = \{ \text{ essential components over } \mathrm{Y} \ \} \tag{5.7}$$

for suitable $m', \tilde{m}' \in \mathbb{N}$, where $1 \leq \tilde{m}' \leq \tilde{m}$ and $\tilde{m}' \leq m' \leq m$. Then

$$\{\mathrm{E}_1, \dots, \mathrm{E}_{\tilde{m}'}\} \subseteq \{ \text{ essential components over } \mathrm{Y} \ \}.$$

Finally, define the two maps

$$\begin{array}{rccc}
\beta : & \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over } \tilde{\mathrm{X}} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over } \mathrm{X} \end{array} \right\} \\
& {\scriptstyle \text{P4.34}} \wr\| & & \wr\| {\scriptstyle \text{P4.34}} \\
& \left\{ \begin{array}{c} \text{essential} \\ \text{components} \\ \text{over } \tilde{\mathrm{Y}} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{essential} \\ \text{components} \\ \text{over } \mathrm{Y} \end{array} \right\} \\
& {\scriptstyle (5.6)} \ \| & & \| \ {\scriptstyle (5.7)} \\
& \{\tilde{\mathrm{E}}_1, \dots, \tilde{\mathrm{E}}_{\tilde{m}'}\} & \longrightarrow & \{\mathrm{E}_1, \dots, \mathrm{E}_{\ell'}\} \\
& \tilde{\mathrm{E}}_j & \longmapsto & \mathrm{E}_j
\end{array} .$$

and

$$\begin{array}{rccc}
\gamma : & \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to } \tilde{\mathrm{X}} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to } \mathrm{X} \end{array} \right\} \\
& \| & & \| \\
& \{\tilde{\mathrm{C}}_i\}_{i \in \tilde{\mathcal{I}}} & \longrightarrow & \{\mathrm{C}_i\}_{i \in \mathcal{I}} \\
& \tilde{\mathrm{C}}_i & \longmapsto & \mathrm{C}_i.
\end{array}$$

Then in particular, suppose that the essential divisor D_v corresponds to an essential component $\tilde{E}_j$ over $\tilde{Y}$ for some $1 \le j \le \tilde{m}'$. Then it will correspond to the essential component E_j over Y (cf. Definition 4.27 and Claim (2)). So let

$$\tilde{\mathcal{N}} : \left\{ \begin{matrix} \text{Nash components} \\ \text{with respect to } \tilde{X} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{essential components} \\ \text{over } \tilde{Y} \end{matrix} \right\}$$

be the Nash map for $\tilde{X}$, which by Lemma 5.32 is bijective since $\tilde{X}$ is affine. Then there exists a Nash component $\tilde{C}_i$ for $i \in \tilde{I}$ such that $\tilde{\mathcal{N}}(\tilde{C}_i) = \tilde{E}_j$.

Combining this fact with the results of Claim (2), Claim (4) and Claim (5), we see that the following diagram commutes for $\tilde{C}_i$:

$$\begin{tikzcd}
\tilde{C}_i \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \tilde{E}_j \arrow[ddd, mapsto] \\
& \left\{ \begin{matrix} \text{Nash components} \\ \text{with respect to } \tilde{X} \end{matrix} \right\} \arrow[r, "\tilde{\mathcal{N}}"] \arrow[d, "\gamma"'] & \left\{ \begin{matrix} \text{essential} \\ \text{components over } \tilde{Y} \end{matrix} \right\} \arrow[d, "\beta"] & \\
& \left\{ \begin{matrix} \text{Nash components} \\ \text{with respect to } X \end{matrix} \right\} \arrow[r, "\mathcal{N}"'] & \left\{ \begin{matrix} \text{essential} \\ \text{components over } Y \end{matrix} \right\} & \\
C_i \arrow[rrr, mapsto] & & & E_j
\end{tikzcd}$$

This shows that we can recover E_j from the Nash map $\mathcal{N}$ for X, which corresponds to the arbitrary essential divisor D_v over X that we picked up at the beginning. Indeed, the diagram is commutative for all Nash components with respect to $\tilde{X}$.

Since our choice of essential divisor over X was arbitrary, we have shown that the Nash map is surjective. The injectivity was already shown in Lemma 4.37, and so $\mathcal{N}$ is bijective. □

Example 5.35. Let

$$e_1 = (1,0,0), \qquad e_2 = (0,1,0), \qquad e_3 = (1,1,c),$$

be points in $N \cong \mathbb{Z}^3$ for some $c > 1$, and set $\sigma := \operatorname{cone}(e_1, e_2, e_3)$. It is clear that every face of σ is smooth, but the whole of σ is not smooth, and hence the toric variety X of σ is singular. Hence $S = N \cap \sigma^\circ$, and the minimal elements of S are points $(1,1,s)$ for $1 \le s < c$. It follows from Theorem 5.34 (actually even from Lemma 5.32) that the number of Nash components of X is equal to the number of essential divisors over X is equal to $c - 1$.

5.6 Comments

There are a few technicalities to iron out, but I believe that the proof of Theorem 5.34 can be completely generalised for any type of variety, with the assumption that the Nash map is bijective in the affine case.

Chapter 6

A counterexample in dimension 3

Our final discussion concerns the recent counterexample offered in [dF13], which shows that the Nash map is not bijective for all varieties of dimension 3. Historically this is one of the more important achievements in the study of the Nash problem, since the technique generalises to any higher dimension as well. The result was the final step in answering the Nash problem with regards to dimension.

Some of the details of this proof are beyond the scope of this thesis. I have tried to give an overview rigorous enough for the reader to appreciate the main ideas, particularly when it comes to the use of arcs. Where detail is lacking, the text has been furnished with references that do give said detail.

In particular, the characterisation of the Nash map is slightly different here. We speak instead of *essential divisorial valuations* as opposed to essential divisors. This is a common approach used in the study of the Nash problem, and therefore it is worth exploring for reasons beyond just this counterexample.

It should be noted that here, we are no longer free to work over fields of arbitrary characteristic.

Assumptions. k is an algebraically closed field of characteristic 0.

6.1 Preliminaries on divisors

Assumptions. X is a normal variety of dimension $n \in \mathbb{N}_0$, and K is the function field of X.

Definition 6.1. The variety X is said to be *factorial* (resp. $\mathbb{Q}$*-factorial*) if every Weil divisor on X is a Cartier (resp. $\mathbb{Q}$-Cartier) divisor.

Lemma 6.2 ([dF13, Lemma 3.1]). *Let $f : Y \to X$ be a resolution of singularities of a normal variety X. If the exceptional set of f has an irreducible component of codimension ≥ 2 in Y, then X is not $\mathbb{Q}$-factorial.*

Definition 6.3 (cf. Definition 4.25).

(a) A *prime divisor on* X is a closed subvariety of codimension 1.

(b) A *Weil divisor on* X is an element of the free abelian group Div X generated by the prime divisors on X. We write such elements as finite sums of the form

$$D = \sum n_i E_i$$

for $n_i \in \mathbb{Z}$ and E_i prime divisors on X.

Given a prime divisor E on X, let $\eta \in E$ be its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring (cf. [Mat86, §11]) and there is a corresponding *discrete valuation* (cf. [Har77, §I.6 and §II.6])

$$\mathrm{ord}_E : K^* \to \mathbb{Z},$$

which has the properties

$$\mathrm{ord}_E(xy) = \mathrm{ord}_E(x) + \mathrm{ord}_E(y)$$
$$\mathrm{ord}_E(x+y) \geq \min\{\mathrm{ord}_E(x), \mathrm{ord}_E(y)\}$$

for all $x, y \in K^* := K \setminus \{0\}$. Specifically, $\mathrm{ord}_E(x)$ is the order of vanishing of $x \in K^*$ along the generic point of E.

Definition 6.4. Let $f \in K^*$. The *divisor of* f is the divisor

$$(f) := \sum_{\substack{E \in \mathrm{Div}\, X \\ E \text{ prime}}} \mathrm{ord}_E(f) E.$$

By [Har77, Lemma I.6.1], this is a finite sum and hence a well-defined divisor. Any divisor of this form is then called a *principal divisor on* X.

Definition 6.5. Two divisors D and D′ on X are said to be *linearly equivalent* if $D - D'$ is a principal divisor. This forms an equivalence relation on the group Div X and the corresponding equivalence classes are called *linear equivalence classes.*

An important example of a linear equivalence class is the *canonical divisor on* X.

Definition 6.6. Suppose that X is also smooth and let $\omega \in \Omega_{X/k}$ be an n-form on X (cf. [Har77, §II.8]), which we can write locally as

$$\omega = f(x_1, \ldots, x_n) \cdot dx_1 \wedge \cdots \wedge dx_n$$

for a suitable rational function f and local coordinates $x_1, \dots, x_n$. Then let (ω) be the divisor of ω on X, which locally is equal to (f). Now take another n-form $\omega' \in \Omega_{X/k}$ which we can write locally as $\omega' = g \cdot \omega$ for suitable g. Then $(\omega') = (g) + (\omega)$, so (ω) and (ω') are linearly equivalent.

The linear equivalence class corresponding to (ω) is called the *canonical divisor on* X, and is denoted K_X.[1]

In fact, the canonical divisor is also defined for singular varieties in the following way: Suppose that X is singular and let $f : Y \to X$ be a resolution of singularities. The pushforward on f induces a bijection between the canonical divisors of Y and the canonical divisors of X (see [dFH09, §3]). Then we say that the canonical divisor K_X is the pushforward of K_Y under f, i.e. $K_X := f_*(K_Y)$.

Definition 6.7. The variety X is said to be $\mathbb{Q}$-*Gorenstein* if the canonical divisor K_X is $\mathbb{Q}$-Cartier, i.e. there exists an $n \in \mathbb{N}$ such that nK_X is a Cartier divisor.

Definition 6.8. Suppose that X is $\mathbb{Q}$-Gorenstein and let $f : Y \to X$ be a birational morphism where Y is normal. Fix a K_Y such that $K_X = f_*(K_X)$. Then the *relative canonical divisor* is

$$K_{Y/X} := K_Y - f^*(K_X),$$

where $f^*(K_X)$ is the pullback of K_X.

The reason that we require X to be $\mathbb{Q}$-Gorenstein is so that we can use the well-defined pullback on Cartier divisors (see [EGA IV$_4$, §21.4]). There exist constructions of the relative canonical divisor for more general varieties (see for example [dFH09]), but the above definition is sufficient for our purposes.

Definition 6.9. Let $f : Y \to X$ be a birational morphism of varieties where Y is normal, and let E be a prime divisor on Y. The *discrepancy of* E *over* X is the integer

$$k_E(X) := \mathrm{ord}_E(K_{Y/X}).$$

Remark 6.10. The discrepancy does not depend on the choice of f in the following sense: Pick up another birational morphism $f : Y' \to X$ with Y' normal, and let E' be a divisor on Y. Then the birational map $Y \to X \dashrightarrow Y'$ is an isomorphism at the generic points of E and E' if, and only if, $\mathrm{ord}_E = \mathrm{ord}_{E'}$. (This also means that they are in the same equivalence class of divisors over X, cf. Definition 4.30.) In this case it is clear that $k_E(X) = k_{E'}(X)$. See also [KM98, Remark 2.23].

[1] Despite the name, the canonical divisor is not actually a divisor, but only a linear equivalence class. Still, it is often treated as a divisor anyway; see [Kol07, §1.6 (1.51)]. One can also say that *a* canonical divisor is just an element of *the* canonical divisor linear equivalence class.

Anyway, it is usually very clear what is meant depending on the use of definite or indefinite article.

6.2 Divisorial valuations

In §4.5 we offered the notion of a divisor *over* a singular variety X (Definition 4.31(a)). We did this by forming a suitable equivalence class of divisors on smooth normal varieties Y that admit resolutions $Y \to X$. Now we are going to form an analogous description using the language described immediately above. We then look at the Nash map in these terms, as is done in [dFD15].

Definition 6.11. Let $f : Y \to X$ be a proper birational morphism of varieties such that Y is normal. Denote the function field of X by K. Given a prime divisor E on Y, we obtain a discrete valuation

$$v := q \cdot \mathrm{val}_E : K^* \to \mathbb{Z},$$

where $q := q(v) \in \mathbb{N}$ is the *multiplicity*[2] *of* v, and where

$$\mathrm{val}_E(h^*) := \mathrm{ord}_E(h \circ f)$$

for each $h^* \in K^*$ that is regular at the generic point of $f(E)$.

A discrete valuation of the form v is said to be a *divisorial valuation on* X.

Divisorial valuations are discussed in detail in [dFEI08].

Remark 6.12. By definition, it follows that prime divisors in the same equivalence class of divisors *over* X (cf. Definition 4.30) define the same divisorial valuation on X.

Definition 6.13. Let $\alpha : \operatorname{Spec} K[[t]] \to X$ be a transversal arc on X. By Proposition 4.11, the ring homomorphism $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ extends to an injective homomorphism of fields $\alpha^* : K(X) \to K((t))$, where K(X) is the function field on X. Then we obtain a discrete valuation

$$\begin{aligned} \mathrm{ord}_\alpha := \mathrm{ord}_t \circ \alpha^* : K(X)^* &\to \mathbb{Z} \\ h^* &\mapsto \mathrm{ord}_t(\alpha^*(h^*)), \end{aligned}$$

and we say that ord_α is the *valuation corresponding to* α.

Now, let $f : Y \to X$ be a resolution of singularities, and let K be the function field of X. Denote by $\{C_i\}_{i \in \mathcal{J}}$ the Nash components with respect to X, and for each $i \in \mathcal{J}$ let $\alpha_i \in C_i$ be the generic point of C_i and let K_i be the function field of C_i. Also, denote the irreducible components of $f^{-1}(\operatorname{Sing} X)$ by $\{E_1, \ldots, E_m\}$.

Since we have field extensions $K_i \supseteq K \subseteq k$, each α_i is an arc of the form

$$\alpha_i : \operatorname{Spec} K_i[[t]] \to X,$$

[2]Not to be confused with the notion of cone multiplicity given in Definition 5.27!

and we obtain a discrete valuation ord_{α_i} on X. Now suppose that $\mathcal{N}(C_i) = \pi_Y^{-1}(E_j)$ for some suitable $1 \leq j \leq \ell$. If we let η be the generic point of E_j, then $f_\infty(\pi_Y^{-1}(\eta)) = \alpha_i$ (the generic point of C_i). We have the following situation:

$$\begin{tikzcd}
\pi_Y^{-1}(\eta) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \alpha_i \arrow[ddd, mapsto] \\
& Y_\infty \arrow[r, "f_\infty"] \arrow[d, "\pi_Y"'] & X_\infty \arrow[d, "\pi_X"] & \\
& Y \arrow[r, "f"'] & X & \\
\eta \arrow[rrr, mapsto] & & & \alpha_i(0)
\end{tikzcd}$$

So $\alpha_i(0)$ is the generic point of $f(E_j)$. Indeed, if E_j is of codimension 1 in Y, we then see that the valuation ord_{α_i} corresponding to α_i is such that

$$\mathrm{ord}_{\alpha_i}(h^*) = \mathrm{ord}_t(\alpha_i^*(h^*)) = \mathrm{ord}_{\alpha_i(0)}(h) = \mathrm{ord}_{f(\eta)}(h) = \mathrm{ord}_\eta(h \circ f) = \mathrm{ord}_{E_j}(h \circ f)$$

for $h^* \in K^*$ regular at $\alpha_i(0)$. Hence ord_{α_i} is the divisorial valuation val_E on X.

Since we can easily pick a suitable resolution f such that $f^{-1}(\mathrm{Sing}\,X)$ is of pure codimension 1 (e.g. by normalization), it follows that all valuations corresponding to (the generic point of) a Nash component are divisorial valuations (cf. Remark 5.6).

Definition 6.14. Consider a divisorial valuation on X. If for every resolution $f : Y \to X$, the center (cf. Definition 4.27) of the corresponding divisor on Y is an irreducible component of $f^{-1}(\mathrm{Sing}\,X)$, then it is said to be an *essential divisorial valuation on* X.

Thus we obtain the correspondence

$$\left\{ \begin{array}{c} \text{essential divisorial} \\ \text{valuations on X} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over X} \end{array} \right\},$$

and so we consider the Nash map as

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{with respect to X} \end{array} \right\} \to \left\{ \begin{array}{c} \text{essential divisorial} \\ \text{valuations on X} \end{array} \right\}.$$

6.3 Some intermediate results

One of the main approaches to the Nash problem involves the use of so-called *wedges* on a singular variety, which were first introduced in [LJ80]. Considering the depth of our discussion, we only need to comprehend the following straightforward definition:

Definition 6.15. Let X be a scheme of finite type over k (here, k could also have positive characteristic). A *wedge on* X is a morphism of the form

$$\Phi : \operatorname{Spec} K[[s,t]] \to X$$

for a field extension K of k.

We now present two lemmas—without proof—which are necessary for our explanation of the counterexample.

Lemma 6.16 ([IK03, Lemma 4.2]). *Let* $Y \subseteq \mathbb{A}_k^{n+1}$ *be a hypersurface with an isolated singularity* P *and suppose that we have a the blow-up* $Z := \mathrm{Bl}_P Y \to Y$ *whose exceptional divisor* E *is a hypersurface in the exceptional divisor* $\mathbb{P}_k^n$ *of the blow-up* $g : \mathrm{Bl}_P \mathbb{A}_k^{n+1} \to \mathbb{A}_k^{n+1}$.

Let $\psi : \operatorname{Spec} k[[t]] \to Z$ *be an arc with contact order* 1 *along* E *and suppose that there exists a line* $L \subseteq E \subseteq \mathbb{P}_k^n$ *through* $\psi(0)$ *such that* $H^1(L, N_{L/E}) = 0$. *Then the arc* $\varphi := g \circ \psi : \operatorname{Spec} k[[t]] \to Y$ *(the* image of ψ on Y*) extends to a smooth wedge*

$$\Phi : \operatorname{Spec} k[[s,t]] \to Y,$$

i.e. we have a commutative diagram

$$\begin{tikzcd} & \operatorname{Spec} k[[s,t]] \arrow[d, "\Phi"] \\ \operatorname{Spec} k[[t]] \arrow[ur, hook] \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

Here, $N_{L/E}$ is the *normal sheaf of* L *in* E (see [Har77, p. 182]) and $H^1(L, N_{L/E})$ is the first cohomology of L with coefficients in the sheaf $N_{L/E}$ (see [Dan96, §2]). Regrettably, to give a satisfactory treatment of these concepts is beyond the scope of this text.

The arc ψ having contact order 1 along E means that the pullback along ψ of the sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_Z$ is the ideal $0 = (t)$ in $k[[t]]$. In particular, $\psi(\eta) \notin E$. See [ELM04].

Lemma 6.17 ([dF13, Lemma 5.2]). *Let* X *be a* 3*-dimensional variety with an isolated singularity at a point we shall denote* O. *Suppose that the blow-up* $\mathrm{Bl}_O X$ *of* X *at* O *is* $\mathbb{Q}$*-factorial and has terminal singularities. Suppose further that the corresponding exceptional divisor of the blow-up is irreducible. Then any essential divisorial valuation* val_E *over* X *such that* $k_E(X) = 1$ *and* $\mathrm{val}_E(\mathfrak{m}_O) = 1$*, is an essential divisorial valuation.*

6.4 The counterexample

Let $\mathbb{A}_k^4 = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ and consider the hypersurface

$$X = \{(x_1^2 + x_2^2 + x_3^2)x_4 + x_1^3 + x_2^3 + x_3^3 + x_4^5 + x_4^6 = 0\} \subseteq \mathbb{A}_k^4.$$

We claim that this is a counterexample to the Nash problem (Problem 4.38).

The hypersurface X has an isolated singularity $O = (0,0,0,0) \in \mathbb{A}^4_k$ of multiplicity 3. First, let's take the blow-up at the maximal ideal $\mathfrak{m}_O$ of O:

$$f : Y := \mathrm{Bl}_O X \to X.$$

The exceptional divisor F of f is then given by

$$F := \{(x_1^2 + x_2^2 + x_3^2)x_4 + x_1^3 + x_2^3 + x_3^3 = 0\} \subseteq \mathbb{P}^3_k,$$

defined in the exceptional divisor $\mathbb{P}^3_k \cong \operatorname{Proj} k[x_1, x_2, x_3, x_4]$ of the blow-up $\mathrm{Bl}_O \mathbb{A}^4_k \to \mathbb{A}^4_k$. Then F has only one singular point, namely $P := (0:0:0:1) \in \mathbb{P}^3_k$. So take the neighbourhood

$$U := \operatorname{Spec} k[u_1, u_2, u_3, x_4] \subseteq \mathrm{Bl}_O \mathbb{A}^4_k,$$

where $u_i := x_i/x_4$ for $i = 1,2,3$. On U we have that $P = (0,0,0,0)$, and Y is defined on U by

$$\{u_1^2 + u_2^2 + u_3^2 + x_4^2 + u_1^3 + u_2^3 + u_3^3 + x_4^3 = 0\}.$$

From this representation we see that P is an *ordinary double point on* Y, i.e. it has multiplicity 2 and distinct tangent directions; see [Har77, §I.5]. Then consider the closure $\overline{Y \cap U}$ taken in $\mathbb{P}^4_k \cong \operatorname{Proj} k[u_0, u_1, u_2, u_3, x_4]$. This is a degree 3 hypersurface inside $\mathbb{P}^4_k$ with a single ordinary double point, and from these facts it is a direct consequence of [Che10, Theorem 1.4] that $\overline{Y \cap U}$ is a factorial variety. It then follows that Y is a factorial variety.

Now take one more blowup

$$g : Z := \mathrm{Bl}_P Y \to Y,$$

which has exceptional divisor E given by

$$E := \{u_1^2 + u_2^2 + u_3^2 + x_4^2 = 0\} \tag{6.1}$$

defined in the exceptional divisor $\mathbb{P}^3_k \cong \operatorname{Proj} k[u_1, u_2, u_3, x_4]$ of the blow-up $\mathrm{Bl}_P U \to U$. In particular, E is smooth and therefore $g : Z \to Y$ and $f \circ g : Z \to X$ are both resolutions. The strategy is to show that E corresponds to an essential divisorial valuation which is not in the image of the Nash map $\mathcal{N}$ (with respect to X).

Claim (1). The divisorial valuation val_E corresponding to E is an essential divisorial valuation on X.

Proof. This will be very brief. We state as fact, and without computation, the following:

$$k_E(X) = 1, \quad k_F(X) = 0, \quad \text{and} \quad \mathrm{val}_E(\mathfrak{m}_O) = 1,$$

where $\mathfrak{m}_O$ is the maximal ideal corresponding to O. (Some additional detail on this can be found in [dF13, Theorem 5.1].) Then val_E satisfies the hypothesis of Lemma 6.17, and is therefore an essential divisorial valuation. ∎

Claim (2). $\mathrm{val}_E \notin \mathrm{Im}(\mathcal{N})$.

Proof. Pick up an arc $\psi : \operatorname{Spec} k[[t]] \to Z$ with contact order 1 along E, and let $\varphi := g \circ \psi : \operatorname{Spec} k[[t]] \to Y$ be its image on Y. Observe from the defining equation (6.1) that $E \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ (cf. [Har77, Exercise I.2.15 and Exercise I.5.12]), so let $L \subseteq E$ be a line through $\psi(0)$ in one of the two rulings of E. Then $N_{L/E} \cong \mathcal{O}_{\mathbb{P}^1_k}$ (see [dF13, Remark 5.3]), where $N_{L/E}$ is the *normal sheaf of* L *in* E (cf. [Har77, p. 182]), and one can conclude that $H^1(L, N_{L/E}) = 0$.

We may then apply Lemma 6.16, which tells us that the arc φ extends to a smooth wedge Φ such that the following diagram commutes:

$$\begin{tikzcd} & \operatorname{Spec} k[[s,t]] \arrow[d, "\Phi"] \\ \operatorname{Spec} k[[t]] \arrow[ur, hook, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

Quickly recall what was said in Remark 3.8, namely that arc spaces are also well-defined on more general schemes than those of finite type over an algebraically closed field. Specifically we are interested in the arc space $(\operatorname{Spec} k[[s,t]])_\infty$ of $\operatorname{Spec} k[[s,t]]$, which is a scheme over k which is *not* of finite type. By [Voj07, Theorem 4.5], we know that this still exists in the sense that it satisfies (3.2) and all the functorial properties that go along with it. We also note that Proposition 3.36 also holds in this case; indeed, the proof is essentially the same as the one given.

Now recall from above that Y is factorial, and hence the exeptional divisor E is a Cartier divisor (cf. Definition 6.1) and the preimage $C := \Phi^{-1}(F)$ is a curve on $\operatorname{Spec} k[[s,t]]$; in particular it is closed and irreducible. For clarity, set $S := \operatorname{Spec} k[[s,t]]$. Since $\varphi(0) \in F$, we note that $\tilde{\varphi}(0) \in C$. We are going to examine the arcs on S through C and conclude that φ is a specialization of some arc through the generic point of F. Treating $\tilde{\varphi}$ as an arc on S, we see that $\tilde{\varphi} \in \pi_S^{-1}(C)$.

Since C is closed and irreducible, it follows from the above remarks[3] that $\pi_S^{-1}(C)$ is a closed irreducible subset of S_∞. Let $\tilde{\alpha}$ be the generic point of $\pi_S^{-1}(C)$ and observe that

$$C \subseteq \pi_S(\overline{\{\tilde{\alpha}\}}) \subseteq \overline{\pi_S(\tilde{\alpha})}.$$

Hence $\tilde{\alpha}$ is an arc on S through the generic point of C, and $\tilde{\varphi} \in \overline{\{\tilde{\alpha}\}}$, i.e. $\tilde{\varphi}$ is a specialization of $\tilde{\alpha}$.

We can transport this specialization to Y_∞ in the following way:

$$\begin{tikzcd} (\operatorname{Spec} k[[s,t]])_\infty \arrow[r, "\Phi_\infty"] \arrow[d, "\pi_S"'] & Y_\infty \arrow[d, "\pi_Y"] \\ \operatorname{Spec} k[[s,t]] \arrow[r, "\Phi"'] & Y \end{tikzcd}$$

[3] Here we are using the more general analogue of Proposition 3.36 on S, which is allowed since S smooth by Lemma 6.16. To be precise, we can just remove the finite type hypothesis from the statement of Proposition 3.36.

We have that

$$\varphi \in \Phi_\infty(\overline{\{\tilde{\alpha}\}}) \subseteq \overline{\Phi_\infty(\tilde{\alpha})} \subseteq Y_\infty,$$

and

$$\alpha := \Phi_\infty(\tilde{\alpha}) = \Phi \circ \tilde{\alpha}$$

is an arc on Y such that $\alpha(0) = \Phi(\tilde{\alpha}(0)) \in F \setminus \{P\}$. (The fact that α is not centered on P is due to the fact that $P = \operatorname{Sing} X$ is closed, and so its preimage $\pi_S^{-1}(\{P\})$ will be closed in C; in particular, this cannot be the generic point of C.) Hence φ is a specialization of α, an arc centered on F but not on P.

But now we see that

$$\begin{aligned} & g_\infty(\pi_Z^{-1}(E)) \subseteq \overline{\pi_Y^{-1}(F \setminus \{P\})} \\ \Longrightarrow \quad & (f \circ g)_\infty(\pi_Z^{-1}(E)) \subseteq f_\infty(\overline{\pi_Y^{-1}(F \setminus \{P\})}) \subseteq \overline{f_\infty(\pi_Y^{-1}(F \setminus \{P\}))}. \end{aligned}$$

In particular, the generic point of $\pi_Z^{-1}(E)$ is not mapped by $(f \circ g)_\infty$ to the generic point of any Nash component with respect to X, and therefore val_E is not in the image of $\mathcal{N}$. ∎

We conclude that the Nash map $\mathcal{N}$ is not bijective for X. The following theorem states the result in more general terms.

Appendices

Appendix A

Categories

$\mathcal{S}\mathrm{et}$	sets
$\mathcal{S}\mathrm{ch}$	schemes
$\mathcal{S}\mathrm{ch}/k$	schemes over a given field k
$\mathcal{S}\mathrm{ch}_{\mathrm{ft}}/k$	schemes of finite type over a given field k
$\mathrm{f}\mathcal{S}\mathrm{ch}/k$	formal schemes over a given field k
$k{-}\mathcal{A}\mathrm{lg}$	k-algebras for a given field k
$[\mathcal{A}, \mathcal{B}]$	functors between two given categories $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{C}^{\mathrm{op}}$	opposite category of a given category $\mathcal{C}$

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